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Multiple D3-instantons and mock modular forms I

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Abstract: We study D3-instanton corrections to the hypermultiplet moduli space in type IIB string theory compactified on a Calabi-Yau threefold. In a previous work, consistency of D3-instantons with S-duality was established at first order in the instanton expansion, using the modular properties of the M5-brane elliptic genus. We extend this analysis to the two-instanton level, where wall-crossing phenomena start playing a role. We focus on the contact potential, an analogue of the Kähler potential which must transform as a modular form under S-duality. We show that it can be expressed in terms of a suitable modification of the partition function of D4-D2-D0 BPS black holes, constructed out of the generating function of MSW invariants (the latter coincide with Donaldson-Thomas invariants in a particular chamber). Modular invariance of the contact potential then requires that, in case where the D3-brane wraps a reducible divisor, the generating function of MSW invariants must transform as a vector-valued mock modular form, with a specific modular completion built from the MSW invariants of the constituents. Physically, this gives a powerful constraint on the degeneracies of BPS black holes. Mathematically, our result gives a universal prediction for the modular properties of Donaldson-Thomas invariants of pure two-dimensional sheaves.

Contents

1. Introduction

The low energy effective action of type II string theory compactified on a Calabi-Yau threefold is determined by the metric on the moduli space, which is a direct product of its vector multiplet and hypermultiplet components. Whereas the former is classically exact, the hypermultiplet moduli space $\mathcal{M}_H$ receives a variety of quantum corrections (see e.g. [1, 2] and references therein). In type IIB string theory, if the volume of the Calabi-Yau threefold $\mathfrak{Y}$ is taken to be large in string units, these quantum corrections can be ordered according to the following hierarchy: i) one-loop and D(-1) instanton corrections, ii) (p, q) string instantons,

iii) D3-instantons, iv) (p,q) five-brane instantons. All these corrections are expected to be governed by topological invariants of $\mathfrak{Y}$, including its intersection form κ_{abc}, Euler number $\chi_{\mathfrak{Y}}$, Chern classes $c_{2,a}$, genus zero Gromov-Witten invariants n_{q_a} and Donaldson-Thomas (DT) invariants $\Omega(\gamma; z^a)$.[1] In addition, they are severely constrained by the fact that the exact metric on $\mathcal{M}_H$ should be quaternion-Kähler [3] and smooth across walls of marginal stability [4, 5] in spite of the discontinuities of the DT invariants $\Omega(\gamma; z^a)$. Most notably, it should carry an isometric action of the modular group $SL(2,\mathbb{Z})$ [6], originating from the S-duality symmetry in uncompactified type IIB string theory.

In order to satisfy the first requirement, it is most convenient to use the twistorial formulation of quaternion-Kähler manifolds [7, 8]. In this framework, quantum corrections to the metric on $\mathcal{M}_H$ are captured by a set of holomorphic functions on the twistor space $\mathcal{Z}$ of $\mathcal{M}_H$, which encode gluing conditions between local Darboux coordinate systems for the canonical complex contact structure on $\mathcal{Z}$. Furthermore, discrete isometries of $\mathcal{M}_H$ must lift to holomorphic coordinate transformations on $\mathcal{Z}$ preserving the contact structure, which constrains the possible gluing conditions. In the presence of a continuous isometry, another important object, central for this work, is the contact potential e^{Φ}, a real function on $\mathcal{M}_H$, defined as the norm of the moment map for the corresponding isometry [8, 9]. Its importance lies in the fact that it provides a Kähler potential on $\mathcal{Z}$, and that it must be invariant under any further discrete isometry, up to a rescaling dictated by the transformation of the contact one-form. In the present context, the isometry corresponds to translation along the NS axion, which is broken only by (p,q) five-brane instantons, while the contact potential determines the 4-dimensional string coupling.

Since the action of the modular group preserves the large volume limit, modular invariance should hold at each level in the aforementioned hierarchy of quantum corrections. For the first two levels, modular invariance was used in [6] to infer the D(-1) and (p,q)-string instanton corrections from the known world-sheet instantons at tree-level and the one-loop correction. The contributions of D3 and D5 instantons were then deduced by requiring symplectic invariance and smoothness across walls of marginal stability [5, 10]. The consistency of D3-instantons with S-duality however depends on special properties of the DT invariants $\Omega(\gamma; z^a)$, where in this case γ labels the charges (p^a, q_a, q_0) of a D3-D1-D(-1) instanton, or more mathematically, the Chern character of a coherent sheaf with support on an effective divisor $\mathcal{D}$ in $\mathfrak{Y}$.

In order to study this problem, it is useful to express the DT invariants $\Omega(\gamma; z^a)$, which in general exhibit wall-crossing behavior with respect to the Kähler moduli z^a, in terms of the so-called Maldacena-Strominger-Witten (MSW) invariants $\Omega^{\rm MSW}(\gamma)$, familiar from the study of the partition function of D4-D2-D0 black holes [11]. Unlike DT invariants, MSW invariants are independent of the moduli. Moreover, in the case where the divisor $\mathcal{D}$ wrapped by the D4-brane is irreducible (in the sense that $\mathcal{D}$ cannot be written as the sum of two effective divisors)[2], the MSW invariants appear as Fourier coefficients of a Jacobi form, namely the elliptic genus $\chi_{\boldsymbol{p}}(\tau, z^a, c^a)$ of the superconformal field theory describing an M5-brane wrapped

[1] Here the index a runs over $1, \dots, b_2(\mathfrak{Y})$, q_a labels effective homology classes $H_2^+(\mathfrak{Y})$, γ labels vectors in the homology lattice $H^{\rm even}(\mathfrak{Y})$, and $z^a = b^a + \mathrm{i} t^a$ are complexified Kähler moduli.

[2] This irreducibility condition has not been fully appreciated in the past, and part of the present work aims at relaxing it.

on $T^2 \times \mathcal{D}$ [11]. More precisely, the elliptic genus decomposes into[3]

$$\chi_{\boldsymbol{p}}(\tau, z^a, c^a) = \sum_{\boldsymbol{\mu}\in\Lambda^\star/\Lambda} h_{\boldsymbol{p},\boldsymbol{\mu}}(\tau)\, \theta_{\boldsymbol{p},\boldsymbol{\mu}}(\tau, t^a, b^a, c^a), \tag{1.1}$$

where $\theta_{\boldsymbol{p},\boldsymbol{\mu}}$ is the Siegel theta series (2.22), a vector-valued modular form of weight $(\frac{b_2-1}{2}, \frac{1}{2})$, and $h_{\boldsymbol{p},\boldsymbol{\mu}}$ is the generating function (2.21) of MSW invariants. When $\mathcal{D}$ is irreducible, $h_{\boldsymbol{p},\boldsymbol{\mu}}$ is a holomorphic vector-valued modular form of weight $(-\frac{b_2}{2}-1, 0)$, so that $\chi_{\boldsymbol{p}}(\tau, z^a, c^a)$ transforms as a Jacobi form of weight $(-\frac{3}{2}, \frac{1}{2})$, as expected from the elliptic genus of a standard SCFT [13, 14, 15].

DT invariants coincide with MSW invariants at the 'large volume attractor point', but in general receive additional contributions proportional to products of MSW invariants with moduli-dependent coefficients, corresponding to black hole bound states [16, 17]. The D3-instanton corrections to the metric can thus be organized as an infinite series in powers of MSW invariants, corresponding to multi-instanton effects. In [12] we considered the one-instanton approximation (and large volume limit) of the D3-instanton corrected metric on $\mathcal{M}_H$, keeping only the first term of the expansion (2.14) of DT invariants in terms of MSW invariants. Relying on the modular properties of MSW invariants encoded in the elliptic genus (1.1), we showed that in this approximation, the metric on $\mathcal{M}_H$ admits an isometric action of the modular group. This result was achieved by showing that S-duality acts on the canonical Darboux coordinates on $\mathcal{Z}$ introduced in [5, 10] by a holomorphic contact transformation. While the transformation properties of Darboux coordinates are, already at the classical level, quite complicated, S-duality requires that the contact potential e^Φ should transform in a simple way, namely as a modular form of weight $(-\frac{1}{2}, -\frac{1}{2})$. In [12] we proved that this is the case by showing that the contact potential is directly related to the elliptic genus (1.1) via the action of a modular covariant derivative.

In this paper, we study the corrections to the metric on $\mathcal{M}_H$ at the two-instanton level, i.e. at order $(\Omega^{\mathrm{MSW}})^2$ in the expansion in powers of MSW invariants. The analysis of the transformation properties of Darboux coordinates and a complete proof of the existence of an isometric action of S-duality on $\mathcal{M}_H$ is deferred to a subsequent paper [18]. In this paper, we shall restrict our attention to the contact potential, which is much simpler but yet encodes all possible quantum corrections.

At two-instanton order, we must take into account both corrections to the contact potential which are quadratic in the DT invariants, and order $(\Omega^{\mathrm{MSW}})^2$ contributions in the relation between DT and MSW invariants. Our main result is as follows: the contact potential can be expressed in terms of the modular covariant derivative of the following BPS partition function

$$\widehat{\mathcal{Z}}_{\boldsymbol{p}} = \sum_{\boldsymbol{\mu}\in\Lambda^\star/\Lambda} \widehat{h}_{\boldsymbol{p},\boldsymbol{\mu}}\, \theta_{\boldsymbol{p},\boldsymbol{\mu}} + \frac{1}{2} \sum_{\boldsymbol{p}_1+\boldsymbol{p}_2=\boldsymbol{p}} \sum_{\boldsymbol{\mu}_i\in\Lambda^\star/\Lambda_i} \widehat{h}_{\boldsymbol{p}_1,\boldsymbol{\mu}_1}\, \widehat{h}_{\boldsymbol{p}_2,\boldsymbol{\mu}_2}\, \widehat{\Psi}_{\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2} + \cdots\,, \tag{1.2}$$

where the dots denote terms of higher order in $\widehat{h}_{\boldsymbol{p},\boldsymbol{\mu}}$. Here two new objects are introduced:

[3]The notations are explained in detail in section 2. Note that our definition of theta series is complex conjugate of the usual one used in [12]. This is to avoid the proliferation of complex conjugated theta functions in our equations and to facilitate comparison with the results of the twistorial formalism.

- $\widehat{\Psi}_{\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}$ is the non-holomorphic theta series constructed in [16] for the lattice of signature $(2, b_2 - 2)$ spanned by the D1-brane charges (q_1, q_2) of the two constituents. It transforms as a vector valued modular form of weight $(b_2 + \frac{1}{2}, \frac{1}{2})$ and captures the wall-crossing dependence of $\widehat{\mathcal{Z}}_{\boldsymbol{p}}$ due to two-center black hole solutions (or equivalently two-centered D3-instantons).

- $\widehat{h}_{\boldsymbol{p},\boldsymbol{\mu}} = h_{\boldsymbol{p},\boldsymbol{\mu}} - \frac{1}{2} R_{\boldsymbol{p},\boldsymbol{\mu}}$, where $R_{\boldsymbol{p},\boldsymbol{\mu}}$ is a non-holomorphic function of τ constructed out of the MSW invariants,

$$R_{\boldsymbol{p},\boldsymbol{\mu}}(\tau) = -\frac{1}{4\pi} \sum_{\boldsymbol{p}_1+\boldsymbol{p}_2=\boldsymbol{p}} \sum_{\boldsymbol{\mu}_i \in \Lambda^\star/\Lambda_i} h_{\boldsymbol{p}_1,\boldsymbol{\mu}_1}(\tau)\, h_{\boldsymbol{p}_2,\boldsymbol{\mu}_2}(\tau) \sum_{\boldsymbol{\rho}\in(\Lambda_1-\tilde{\mu})\cap(\Lambda_2+\tilde{\mu})} (-1)^{\mathrm{S}_{\boldsymbol{p}_1,\boldsymbol{p}_2}(\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\boldsymbol{\rho})}$$
$$\times \left|\mathrm{S}_{\boldsymbol{p}_1,\boldsymbol{p}_2}(\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\boldsymbol{\rho})\right| \beta_{\frac{3}{2}}\left(\frac{2\tau_2\left(\mathrm{S}_{\boldsymbol{p}_1,\boldsymbol{p}_2}(\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\boldsymbol{\rho})\right)^2}{(pp_1p_2)}\right) e^{\pi \mathrm{i}\tau \mathrm{Q}_{\boldsymbol{p}_1,\boldsymbol{p}_2}(\boldsymbol{\nu}_1,\boldsymbol{\nu}_2)}, \qquad (1.3)$$

  where $\beta_{\frac{3}{2}}$ is the function defined in (2.29) and the definitions of other notations can be found in Appendix B.

When the effective divisor $\mathcal{D}$ is irreducible, the sum over $\boldsymbol{p}_1, \boldsymbol{p}_2$ is empty so that $R_{\boldsymbol{p},\boldsymbol{\mu}}$ and the second term in (1.2) vanish and $\widehat{\mathcal{Z}}_{\boldsymbol{p}}$ reduces to the elliptic genus (1.1). If on the contrary $\mathcal{D}$ can be decomposed into a sum of two effective divisors $\mathcal{D}_1 + \mathcal{D}_2$, then modular invariance of the contact potential requires that the non-holomorphic function $\widehat{h}_{\boldsymbol{p},\boldsymbol{\mu}}$ must transform as a (vector-valued) modular form of weight $(-\frac{b_2}{2} - 1, 0)$. This shows that the holomorphic generating function $h_{\boldsymbol{p},\boldsymbol{\mu}}$ is not a modular form, but rather a (mixed) mock modular form [19, 20].

A similar modular anomaly was in fact observed long ago for the partition function of topologically twisted $\mathcal{N} = 4$ Yang-Mills theory with gauge group $U(2)$ on a complex surface in [21] and, more recently, in [22]. This set-up was related to the case of multiple M5-branes wrapped on a rigid divisor in a non-compact threefold in [23, 24]. For M5-branes wrapped on non-rigid divisors in an elliptically fibered compact threefold, such an anomaly was also argued to appear in [25] using the holomorphic anomaly in topological string theory [26] and T-duality. However in the latter context the anomaly is of quasi-modular type rather then mock-modular.

Modular or holomorphic anomalies are also known to occur in the context of quantum gravity partition functions for $\mathrm{AdS}_3/\mathrm{CFT}_2$ [27], non-compact coset conformal field theories [28], and partition functions for BPS black holes in $\mathcal{N} = 4$ supergravity [29]. In the context of black hole partition functions, the non-holomorphic completion was related to the spectral anomaly in the continuum of scattering states in [30]. Our result shows that modular or holomorphic anomalies generally affect M5-branes or D4-branes wrapped on reducible divisors in an arbitrary compact Calabi-Yau threefold, and gives a precise prediction for the modular completion in the case where $\mathcal{D}$ is the sum of two irreducible divisors. Physically, this gives a powerful constraint on the degeneracies of D4-D2-D0 brane black holes composed of two D4-branes. In particular, the mock modularity of $h_{\boldsymbol{p},\boldsymbol{\mu}}$ affects the asymptotic growth of the degeneracies [31]. Mathematically, upon re-expressing the MSW invariants in terms of DT-invariants, our result gives a universal prediction for the modularity of DT invariants for pure 2-dimensional sheaves, which is receiving increasing attention from the mathematics community, see e.g. [32, 33, 34, 35, 36]. Using similar techniques, it should be possible in principle to determine the modular anomaly in the case where $\mathcal{D}$ can split into a sum of more than two irreducible divisors.

The organization of the paper is as follows. In section 2 we discuss the BPS invariants counting D3-brane instantons and associated modular forms. In section 3, we review the twistorial formulation of the D-instanton corrected hypermultiplet moduli space of type IIB string theory compactified on a Calabi-Yau threefold. Then in section 4 we compute the D3-instanton contribution to the contact potential in the two-instanton approximation and express it in terms of $\widehat{\mathcal{Z}}_{\boldsymbol{p}}$. Finally, we conclude in section 5. Appendices A, B and C contain some useful material and details of our calculations.

2. BPS invariants for D3-instantons and mock modularity

In this section, we discuss the modular properties of the BPS invariants which control D3-brane instanton corrections to the hypermultiplet moduli space $\mathcal{M}_H$ in type IIB string theory compactified on a Calabi-Yau threefold $\mathfrak{Y}$. The same invariants also control the degeneracies of D4-D2-D0 black holes in type IIA string theory compactified on the same threefold $\mathfrak{Y}$. When the D3-brane wraps a primitive effective divisor[4] $\mathcal{D}$, these invariants are claimed to be Fourier coefficients of a vector-valued modular form. Instead, we will argue that, when $\mathcal{D} = \sum_{i=1}^{n} \mathcal{D}_i$ is the sum of n irreducible divisors, the invariants are the coefficients of the *holomorphic part* of a real-analytic modular form. For $n = 2$, we show that this holomorphic part is in fact a *mixed mock* modular form, whose modular anomaly is controlled by the invariants associated to $\mathcal{D}_i$.

2.1 D3-instantons, DT and MSW invariants

Let us first introduce some mathematical objects and notations relevant for D3-instantons. As in [12], we denote by γ_a an integer irreducible basis of $\Lambda = H_4(\mathfrak{Y}, \mathbb{Z})$, ω_a their Poincaré dual 2-forms, γ^a an integer basis of $\Lambda^* = H_2(\mathfrak{Y}, \mathbb{Z})$, ω^a their Poincaré dual 4-forms, and $\omega_{\mathfrak{Y}}$ the volume form of $\mathfrak{Y}$ such that

$$\omega_a \wedge \omega_b = \kappa_{abc}\, \omega^c, \qquad \omega_a \wedge \omega^b = \delta_a^b\, \omega_{\mathfrak{Y}}, \qquad \int_{\gamma^a} \omega_b = \int_{\gamma_b} \omega^a = \delta_b^a, \tag{2.1}$$

where κ_{abc} is the intersection form, integer-valued and symmetric in its indices. For brevity we shall denote $(lkp) = \kappa_{abc} l^a k^b p^c$ and $(kp)_a = \kappa_{abc} k^b p^c$. We introduce furthermore the Kähler cone as the set of $d^a \omega_a \in H^2(\mathfrak{Y}, \mathbb{R})$ such that

$$d^3 \geq 0, \qquad (rd^2) \geq 0, \qquad k_a d^a \geq 0, \tag{2.2}$$

for all effective divisors $r^a \gamma_a \in H_4^+(\mathfrak{Y}, \mathbb{Z})$ (i.e. $r^a \geq 0$ for all a, not all of them vanishing simultaneously) and effective curves $k_a \gamma^a \in H_2^+(\mathfrak{Y}, \mathbb{Z})$.

[4] We will use the following specifications for a divisor $\mathcal{D}$. See for more details for example [37]. A divisor γ is *irreducible*, if γ is an irreducible analytic hypersurface of $\mathfrak{Y}$. An *effective* divisor $\mathcal{D}$ is a finite linear combination $\mathcal{D} = \sum_a r^a \gamma_a$ of irreducible divisors γ_a with $r^a \in \mathbb{N}_0$ for all a. We call a divisor *primitive* if the gcd($\{r^a\}$)=1.

A D3-instanton is described by a coherent sheaf $\mathcal{E}$ of rank r supported on a divisor $\mathcal{D} \subset \mathfrak{Y}$. The homology class of the divisor $\mathcal{D}$ may be expanded on the basis of 4-cycles as $\mathcal{D} = d^a\gamma_a$. The D-brane charges are given by the components of the generalized Mukai vector of $\mathcal{E}$ on a basis of $H^{\rm even}(\mathfrak{Y}, \mathbb{Z})$,

$$\gamma = \mathrm{ch}\,\mathcal{E}\,\sqrt{\mathrm{Td}\,\mathfrak{Y}} = p^a\omega_a - q_a\omega^a + q_0\,\omega_{\mathfrak{Y}}\,, \tag{2.3}$$

where $p^a = rd^a$. The charges p^a, q_a, q_0 satisfy the following quantization conditions[5]

$$p^a \in \mathbb{Z}, \qquad q_a \in \mathbb{Z} + \frac{1}{2}\,\kappa_{abc}p^bp^c, \qquad q_0 \in \mathbb{Z} - \frac{1}{24}\,p^a c_{2,a}. \tag{2.4}$$

We denote the corresponding charge lattice by Γ, and its intersection with the Kähler cone (2.2) by Γ_+. Upon tensoring the sheaf $\mathcal{E}$ with a line bundle $\mathcal{L}$ on $\mathcal{D}$, with $c_1(\mathcal{L}) = -\epsilon^a\omega_a$, the magnetic charge p^a is invariant, while the electric charges q_a, q_0 vary by a 'spectral flow'

$$q_a \mapsto q_a - \kappa_{abc}p^b\epsilon^c, \qquad q_0 \mapsto q_0 - \epsilon^a q_a + \frac{1}{2}\,\kappa_{abc}p^a\epsilon^b\epsilon^c. \tag{2.5}$$

This transformation leaves invariant the combination

$$\hat{q}_0 \equiv q_0 - \frac{1}{2}\,\kappa^{ab}q_aq_b, \tag{2.6}$$

where κ^{ab} is the inverse of $\kappa_{ab} = \kappa_{abc}p^c$, a quadratic form of signature $(1, b_2-1)$ on $\Lambda\otimes\mathbb{R} \simeq \mathbb{R}^{b_2}$. We use this quadratic form to identify $\Lambda \otimes \mathbb{R}$ and $\Lambda^* \otimes \mathbb{R}$, and use bold-case letters to denote the corresponding vectors. We also identify Λ with its image in Λ^*. Note however that the map $\epsilon^a \mapsto \kappa_{ab}\epsilon^b$ is in general not surjective: the quotient Λ^*/Λ is a finite group of order $|\det\kappa_{ab}|$. The transformation (2.5) preserves the residue class $\mu_a \in \Lambda^*/\Lambda$ defined by

$$q_a = \mu_a + \frac{1}{2}\,\kappa_{abc}p^bp^c + \kappa_{abc}p^b\epsilon^c, \qquad \boldsymbol{\epsilon} \in \Lambda\,. \tag{2.7}$$

We note also that the invariant charge $\hat{q}_0$ is bounded from above by $\hat{q}_0^{\rm max} = \frac{1}{24}(p^3 + c_{2,a}p^a)$.

The contribution of a single D3-instanton to the metric on $\mathcal{M}_H$ is proportional to the DT invariant $\Omega(\gamma; \boldsymbol{z})$, which is the (weighted) Euler characteristic of the moduli space of semi-stable sheaves with fixed Mukai vector γ. The relevant stability condition is Π-stability [39], which reduces to slope stability in the large volume limit. The latter stability condition states that for each subsheaf $\mathcal{E}'(\gamma') \subset \mathcal{E}(\gamma)$ the following inequality is satisfied

$$\frac{(q'_a + (bp')_a)t^a}{(p't^2)} \leq \frac{(q_a + (bp)_a)t^a}{(pt^2)}. \tag{2.8}$$

It is useful to define the rational DT invariant [40, 41, 42],

$$\bar{\Omega}(\gamma; \boldsymbol{z}) = \sum_{d|\gamma} \frac{1}{d^2}\,\Omega(\gamma/d; \boldsymbol{z})\,, \tag{2.9}$$

which reduces to the integer-valued DT invariant $\Omega(\gamma; \boldsymbol{z})$ when γ is a primitive vector, but is in general rational-valued. Both Ω and $\bar{\Omega}$ are piecewise constant as a function of the complexified

[5]The electric charges q_a and q_0 (denoted by q'_a, q'_0 in [38]) are not integer valued. They are related to the integer charges which appear naturally on the type IIA side by a rational symplectic transformation [38].

Kähler moduli $z^a = b^a + \mathrm{i}t^a$, but are discontinuous across walls of marginal stability where the sheaf becomes unstable, i.e. the codimension-one subspaces of the Kähler cone across which the inequality (2.8) flips. Ω and $\bar{\Omega}$ are in general *not* invariant under the spectral flow (2.5), but they are invariant under the combination of (2.5) with a compensating shift of the Kalb-Ramond field, $b^a \mapsto b^a + \epsilon^a$.

A physical way to understand the moduli dependence of $\Omega(\gamma; \boldsymbol{z})$ is to note that the same invariant counts D4-D2-D0 brane bound states in type IIA theory compactified on the same CY threefold $\mathfrak{Y}$. The mass of a single-particle BPS state is equal to the modulus of the central charge function $Z_\gamma = q_\Lambda z^\Lambda - p^\Lambda F_\Lambda(z)$ (where $\Lambda = (0, a) = 0, \dots, b_2$ and $F_\Lambda = \partial_{X^\Lambda} F(X)$ is the derivative of the holomorphic prepotential F). Some of these single-particle BPS states may however arise as bound states of more elementary constituents with charge γ_i such that $\sum_i \gamma_i = \gamma$. Typically, these bound states exist only in some chamber in Kähler moduli space, and decay across walls of marginal stability where the central charges $Z(\gamma_i)$ become aligned, so that the mass $|Z_\gamma|$ coincides with the sum $\sum_i |Z_{\gamma_i}|$ of the masses of the constituents. A similar picture exists for D3-instantons, where the modulus of the central charge controls the classical action, but the analogue of the notion of single-particle state is somewhat obscure.

At the special value of the moduli $\boldsymbol{z}(\gamma) = -\boldsymbol{q} + \mathrm{i}\boldsymbol{p}$ given by the attractor mechanism, no bound states exist, and therefore $\Omega(\gamma; \boldsymbol{z}(\gamma))$ counts elementary states, which cannot decay. Since we are only interested in the large volume limit, we define the 'MSW invariants' $\Omega^{\mathrm{MSW}}(\gamma) = \Omega(\gamma; \boldsymbol{z}_\infty(\gamma))$ as the DT invariants evaluated at the large volume attractor point,

$$\boldsymbol{z}_\infty(\gamma) = \lim_{\lambda \to +\infty} \left(\boldsymbol{b}(\gamma) + \mathrm{i}\lambda \boldsymbol{t}(\gamma)\right) = \lim_{\lambda \to +\infty} \left(-\boldsymbol{q} + \mathrm{i}\lambda \boldsymbol{p}\right). \tag{2.10}$$

The reason for the name MSW (Maldacena-Strominger-Witten) is that when $\boldsymbol{p}$ corresponds to a very ample primitive divisor, these states are in fact described by the superconformal field theory discussed in [11]. It is important that, due to the symmetry (2.5), $\Omega^{\mathrm{MSW}}(\gamma)$ only depend on p^a, μ_a and $\hat{q}_0$ defined in (2.6) and (2.7). We shall therefore write $\Omega^{\mathrm{MSW}}(\gamma) = \Omega^{\mathrm{MSW}}_{\boldsymbol{p},\boldsymbol{\mu}}(\hat{q}_0)$.

Away from the large volume attractor point (but still in the large volume limit), the DT invariant $\Omega(\gamma; \boldsymbol{z})$ receives additional contributions from bound states with charges $\gamma_i = (0, p_i^a, q_{i,a}, q_{i,0}) \in \Gamma_+$ such that $\sum_i \gamma_i = \gamma$ and $\boldsymbol{p}_i \neq 0$ for each i. For $n = 2$, the case of primary interest in this work, bound states exist if and only if the sign of $\mathrm{Im}(Z_{\gamma_1} \bar{Z}_{\gamma_2})$ is equal to the sign of $\langle \gamma_1, \gamma_2 \rangle = p_2^\Lambda q_{1,\Lambda} - p_1^\Lambda q_{2,\Lambda}$ [43]. In the large volume limit, one has

$$\mathrm{Im}(Z_{\gamma_1} \bar{Z}_{\gamma_2}) = -\frac{1}{2} \sqrt{(p_1 t^2)\,(p_2 t^2)\,(p t^2)}\, \mathcal{I}_{\gamma_1 \gamma_2}, \tag{2.11}$$

where

$$\mathcal{I}_{\gamma_1 \gamma_2} = \frac{(p_2 t^2)\,(q_{1,a} + (b p_1)_a)\, t^a - (p_1 t^2)\,(q_{2,a} + (b p_2)_a)\, t^a}{\sqrt{(p_1 t^2)\,(p_2 t^2)\,(p t^2)}} \tag{2.12}$$

is invariant under rescaling of t^a. It is convenient to define the 'sign factor'

$$\Delta^{\boldsymbol{t}}_{\gamma_1 \gamma_2} = \frac{1}{2} \Big(\mathrm{sgn}(\mathcal{I}_{\gamma_1 \gamma_2}(\boldsymbol{t})) - \mathrm{sgn}(\langle \gamma_1, \gamma_2 \rangle) \Big), \tag{2.13}$$

where we indicated explicitly the dependence on the Kähler moduli. This factor takes the value ± 1 when bound states are allowed, or 0 otherwise. The DT invariants are then expressed in terms of the MSW invariants by [16]

$$\bar\Omega(\gamma;\boldsymbol{z}) = \bar\Omega^{\rm MSW}(\gamma) + \frac{1}{2}\sum_{\substack{\gamma_1,\gamma_2\in\Gamma_+\\ \gamma_1+\gamma_2=\gamma}} (-1)^{\langle\gamma_1,\gamma_2\rangle}\langle\gamma_1,\gamma_2\rangle\Delta^{\boldsymbol{t}}_{\gamma_1\gamma_2}\bar\Omega^{\rm MSW}(\gamma_1)\,\bar\Omega^{\rm MSW}(\gamma_2) + \cdots\,, \tag{2.14}$$

where the dots denote contributions of higher order in the MSW invariants.

2.2 Modularity of the BPS partition function

Let us now consider the partition function of DT invariants with fixed magnetic charge $\boldsymbol{p}$. Let $\tau = \tau_1 + i\tau_2 \in \mathbb{H}$, $\boldsymbol{c} \in \mathbb{R}^{b_2}$ the RR potentials conjugate to D1-brane charges, and $\boldsymbol{b} \in \mathbb{R}^{b_2}$ the Kalb-Ramond field. The BPS partition function is defined as the following generating function of DT-invariants

$$\mathcal{Z}_{\boldsymbol{p}}(\tau,\boldsymbol{z},\boldsymbol{c}) = e^{\pi\tau_2(pt^2)}\sum_{q_\Lambda}\bar\Omega(\gamma;\boldsymbol{z})\,(-1)^{\boldsymbol{p}\cdot\boldsymbol{q}}\,e^{-2\pi\tau_2|Z_\gamma|-2\pi\mathrm{i}\tau_1\left(q_0+\boldsymbol{b}\cdot\boldsymbol{q}+\frac12\boldsymbol{b}^2\right)+2\pi\mathrm{i}\boldsymbol{c}\cdot\left(\boldsymbol{q}+\frac12\boldsymbol{b}\right)}, \tag{2.15}$$

where the sum goes over charges satisfying the quantization conditions (2.4). The DT-invariants are weighted by the Boltzmann factor $\exp(-2\pi\tau_2|Z_\gamma|)$ and by a phase factor induced by the couplings of the charges to the potentials τ_1, $\boldsymbol{b}$ and $\boldsymbol{c}$. The factor $(-1)^{\boldsymbol{p}\cdot\boldsymbol{q}}$ is motivated by modular properties of $\mathcal{Z}_{\boldsymbol{p}}$, whereas the prefactor $e^{\pi\tau_2(pt^2)}$ is included so as to subtract the leading divergent term in the large volume limit of $|Z_\gamma|$:

$$|Z_\gamma| = \frac{1}{2}(pt^2) - q_0 + (q+b)_+^2 - (\boldsymbol{q}+\frac{1}{2}\boldsymbol{b})\cdot\boldsymbol{b} + \cdots\,. \tag{2.16}$$

Here the dots denote terms of order $1/(pt^2)$ and, as in [12], we defined

$$\boldsymbol{q}_+ = \frac{q_a t^a}{(pt^2)}\,\boldsymbol{t}\,, \qquad \boldsymbol{q}_- = \boldsymbol{q} - \boldsymbol{q}_+\,, \qquad q_+ = \frac{q_a t^a}{\sqrt{(pt^2)}}\,, \tag{2.17}$$

so that $q_+^2 = (\boldsymbol{q}_+)^2 = \boldsymbol{q}^2 - (\boldsymbol{q}_-)^2$. In the following we shall study the behavior of the BPS partition function (2.15) under modular transformations.

Substituting (2.14) into (2.15), one obtains an expansion in powers of the MSW invariants

$$\mathcal{Z}_{\boldsymbol{p}}(\tau,\boldsymbol{z},\boldsymbol{c}) = \sum_{n\ge1}\mathcal{Z}^{(n)}_{\boldsymbol{p}}(\tau,\boldsymbol{z},\boldsymbol{c}), \tag{2.18}$$

where $\mathcal{Z}^{(n)}_{\boldsymbol{p}}$ corresponds to the terms of degree n in $\bar\Omega^{\rm MSW}(\gamma_i)$. Due to the symmetry of the MSW invariants under the spectral flow (2.5), all terms in this expansion have a theta series decomposition. Indeed, decomposing the vectors $\boldsymbol{q}_i$ according to (2.7), we find, for the first [13, 14, 15] and second [16] terms

$$\mathcal{Z}^{(1)}_{\boldsymbol{p}}(\tau,\boldsymbol{z},\boldsymbol{c}) = \chi_{\boldsymbol{p}}(\tau,\boldsymbol{z},\boldsymbol{c}) = \sum_{\boldsymbol{\mu}\in\Lambda^\star/\Lambda} h_{\boldsymbol{p},\boldsymbol{\mu}}(\tau)\,\theta_{\boldsymbol{p},\boldsymbol{\mu}}(\tau,\boldsymbol{t},\boldsymbol{b},\boldsymbol{c}), \tag{2.19}$$

$$\mathcal{Z}^{(2)}_{\boldsymbol{p}}(\tau,\boldsymbol{z},\boldsymbol{c}) = \frac{1}{2}\sum_{\boldsymbol{p}_1+\boldsymbol{p}_2=\boldsymbol{p}}\ \sum_{\boldsymbol{\mu}_i\in\Lambda^\star/\Lambda_i} h_{\boldsymbol{p}_1,\boldsymbol{\mu}_1}(\tau)\,h_{\boldsymbol{p}_2,\boldsymbol{\mu}_2}(\tau)\,\Psi_{\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}(\tau,\boldsymbol{t},\boldsymbol{b},\boldsymbol{c}). \tag{2.20}$$

Here, we denote by Λ_i the image of Λ inside Λ^* under the map $\epsilon^a \mapsto \kappa_{abc}\epsilon^b p_i^c$ and introduce the following objects (we denote $\mathbf{E}(x) = e^{2\pi\mathrm{i}x}$):

- a holomorphic function of the modular parameter τ built from the MSW invariants

$$h_{\boldsymbol{p},\boldsymbol{\mu}}(\tau)=\sum_{\hat q_0\le \hat q_0^{\max}} \bar\Omega^{\rm MSW}_{\boldsymbol{p},\boldsymbol{\mu}}(\hat q_0)\,\mathbf{E}(-\hat q_0\tau)\,; \tag{2.21}$$

- the Siegel-Narain theta series

$$\theta_{\boldsymbol{p},\boldsymbol{\mu}}(\tau,\boldsymbol{t},\boldsymbol{b},\boldsymbol{c})=\sum_{\boldsymbol{k}\in\Lambda+\boldsymbol{\mu}+\frac12\boldsymbol{p}} (-1)^{\boldsymbol{k}\cdot\boldsymbol{p}}\,\mathcal{X}^{(\theta)}_{\boldsymbol{p},\boldsymbol{k}}\,, \tag{2.22}$$

  where

$$\mathcal{X}^{(\theta)}_{\boldsymbol{p},\boldsymbol{k}}=\mathbf{E}\Big(-\frac{\tau}{2}\,(\boldsymbol{k}+\boldsymbol{b})_-^2-\frac{\bar\tau}{2}\,(\boldsymbol{k}+\boldsymbol{b})_+^2+\boldsymbol{c}\cdot(\boldsymbol{k}+\tfrac12\boldsymbol{b})\Big)\,; \tag{2.23}$$

- the 'double theta series' [16]

$$\Psi_{\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}(\tau,\boldsymbol{t},\boldsymbol{b},\boldsymbol{c})=\sum_{\boldsymbol{k}_i\in\Lambda_i+\boldsymbol{\mu}_i+\frac12\boldsymbol{p}_i} (-1)^{\boldsymbol{p}_1\cdot\boldsymbol{k}_1+\boldsymbol{p}_2\cdot\boldsymbol{k}_2+(p_1^2p_2)}\langle\gamma_1,\gamma_2\rangle\,\Delta^{\boldsymbol{t}}_{\gamma_1\gamma_2}e^{2\pi\tau_2\mathcal{I}^2_{\gamma_1\gamma_2}}\mathcal{X}^{(\theta)}_{\boldsymbol{p}_1,\boldsymbol{k}_1}\mathcal{X}^{(\theta)}_{\boldsymbol{p}_2,\boldsymbol{k}_2}, \tag{2.24}$$

  where $\mathcal{I}_{\gamma_1\gamma_2}$ and $\Delta^{\boldsymbol{t}}_{\gamma_1\gamma_2}$ are defined in (2.12) and (2.13).

The theta series decompositions (2.19) and (2.20) provide the starting point to discuss the modular properties of the BPS partition function. The modular group acts by the following transformations

$$\tau\mapsto\frac{a\tau+b}{c\tau+d}\,,\qquad \boldsymbol{t}\mapsto|c\tau+d|\,\boldsymbol{t},\qquad \begin{pmatrix}\boldsymbol{c}\\ \boldsymbol{b}\end{pmatrix}\mapsto\begin{pmatrix}a&b\\ c&d\end{pmatrix}\begin{pmatrix}\boldsymbol{c}\\ \boldsymbol{b}\end{pmatrix}, \tag{2.25}$$

with $ad-bc=1$. Under this action, the theta series $\theta_{\boldsymbol{p},\boldsymbol{\mu}}$ is well-known to transform as a vector-valued Jacobi form of weight $(\frac{b_2-1}{2},\frac12)$ and multiplier system M_θ. In contrast, the double theta series (2.24) does *not* transform as a vector-valued Jacobi form under $SL(2,\mathbb{Z})$. However, it was shown in [16], using similar techniques as in [19], that it can be completed into a vector-valued modular form $\widehat\Psi=\Psi+\Psi^{(+)}+\Psi^{(-)}$ of weight $(b_2+\frac12,\frac12)$, at the expense of adding two double theta series of the form

$$\Psi^{(\pm)}_{\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}(\tau,\boldsymbol{t},\boldsymbol{b},\boldsymbol{c})=\sum_{\boldsymbol{k}_i\in\Lambda_i+\boldsymbol{\mu}_i+\frac12\boldsymbol{p}_i} (-1)^{\boldsymbol{p}_1\cdot\boldsymbol{k}_1+\boldsymbol{p}_2\cdot\boldsymbol{k}_2+(p_1^2p_2)}\,\Pi^{(\pm)}_{\gamma_1\gamma_2}\,e^{2\pi\tau_2\mathcal{I}^2_{\gamma_1\gamma_2}}\mathcal{X}^{(\theta)}_{\boldsymbol{p}_1,\boldsymbol{k}_1}\mathcal{X}^{(\theta)}_{\boldsymbol{p}_2,\boldsymbol{k}_2}, \tag{2.26}$$

where the insertions are given by the following expressions

$$\Pi^{(+)}_{\gamma_1\gamma_2}=\sqrt{\frac{(pt^2)\,(p_1p_2t)^2}{8\pi^2\tau_2\,(p_1t^2)\,(p_2t^2)}}\,e^{-2\pi\tau_2\mathcal{I}^2_{\gamma_1,\gamma_2}}-\frac12\,\langle\gamma_1,\gamma_2\rangle\,\mathrm{sgn}(\mathcal{I}_{\gamma_1\gamma_2})\,\beta_{\frac12}\big(2\tau_2\mathcal{I}^2_{\gamma_1\gamma_2}\big)\,, \tag{2.27}$$

$$\Pi^{(-)}_{\gamma_1\gamma_2}=-\frac{1}{4\pi}|\langle\gamma_1,\gamma_2\rangle|\,\beta_{\frac32}\left(\frac{2\tau_2\langle\gamma_1,\gamma_2\rangle^2}{(pp_1p_2)}\right). \tag{2.28}$$

Here we used the function $\beta_\nu(y)=\int_y^{+\infty}\mathrm{d}u\,u^{-\nu}e^{-\pi u}$, so that for $x\in\mathbb{R}$

$$\beta_{\frac12}(x^2)=\mathrm{Erfc}(\sqrt{\pi}|x|),\qquad \beta_{\frac32}(x^2)=2|x|^{-1}e^{-\pi x^2}-2\pi\beta_{\frac12}(x^2). \tag{2.29}$$

In Appendix A we provide a simple proof of the modular invariance of $\widehat{\Psi}$ based on Vignéras' theorem [44]. For the proof, it is important that the insertions $\Pi^{(\pm)}_{\gamma_1\gamma_2}$ cancel the discontinuities of the sign factor $\Delta^{\boldsymbol{t}}_{\gamma_1\gamma_2}$ in (2.24) on the loci $\mathcal{I}_{\gamma_1,\gamma_2}=0$ or $\langle\gamma_1,\gamma_2\rangle=0$ in the $2b_2$-dimensional space spanned by the vectors $(\boldsymbol{k}_1,\boldsymbol{k}_2)$, so that the summand in the completed theta series $\widehat{\Psi}$ is a smooth function. It is also important to remark that both functions (2.26) are exponentially suppressed as $\tau_2\to+\infty$, and that $\Pi^{(-)}_{\gamma_1\gamma_2}$ is independent of the Kähler moduli, whereas $\Pi^{(+)}_{\gamma_1\gamma_2}$ does depend on t^a through $\mathcal{I}_{\gamma_1,\gamma_2}$ defined in (2.12).

In [13, 14, 15, 16, 12], it was argued that the first term (2.19) transforms as a modular form of weight $(-\frac{3}{2},\frac{1}{2})$. As a consequence, the generating function $h_{\boldsymbol{p},\boldsymbol{\mu}}$ had to transform as a vector-valued modular form of weight $(-\frac{b_2}{2}-1,0)$ and multiplier system $M(g)=M_{\mathcal{Z}}\times M_\theta^{-1}$, where $M_{\mathcal{Z}}=e^{2\pi\mathrm{i}\epsilon(g)p^a c_{2,a}}$ and $\epsilon(g)$ is the multiplier system of the Dedekind eta function. This proposal has been confirmed in examples where the effective divisor $\mathcal{D}$ wrapped by the D3-brane is irreducible [13, 45], but its validity for a general non-primitive or reducible divisor remained to be assessed.

In fact, the example of N D4-branes in non-compact Calabi-Yau manifolds (or equivalently topologically twisted $\mathcal{N}=4$ $U(N)$ Yang-Mills theory [21]) indicates that $h_{\boldsymbol{p},\boldsymbol{\mu}}$ is unlikely to be modular in general. In the context of $\mathcal{N}=4$ Yang-Mills, examples are known with $\boldsymbol{p}$ reducible (more precisely, with the gauge group $U(2)$ [21, 23, 22]), where $h_{\boldsymbol{p},\boldsymbol{\mu}}$ is not modular, but becomes so after adding to it a suitable non-holomorphic function $R_{\boldsymbol{p},\boldsymbol{\mu}}$. In other words,

$$\widehat{h}_{\boldsymbol{p},\boldsymbol{\mu}}(\tau)=h_{\boldsymbol{p},\boldsymbol{\mu}}(\tau)-\frac{1}{2}\,R_{\boldsymbol{p},\boldsymbol{\mu}}(\tau) \tag{2.30}$$

is a vector-valued modular form at the cost of being non-holomorphic, while $h_{\boldsymbol{p},\boldsymbol{\mu}}$ is a vector-valued (mixed) mock modular form [19, 20]. Given that additional non-holomorphic terms were also required to turn the double theta series (2.24) into a modular form $\widehat{\Psi}$, we expect that for a general divisor, the holomorphic generating function of MSW invariants $h_{\boldsymbol{p},\boldsymbol{\mu}}$ will only become modular after the addition of a suitable non-holomorphic function.

Assuming then that $R_{\boldsymbol{p},\boldsymbol{\mu}}$ exists such that (2.30) is a vector-valued modular form, the modular completion of the BPS partition function (2.15) becomes

$$\widehat{\mathcal{Z}}_{\boldsymbol{p}}(\tau,\boldsymbol{z},\boldsymbol{c})=\sum_{n\geq 1}\widehat{\mathcal{Z}}^{(n)}_{\boldsymbol{p}}(\tau,\boldsymbol{z},\boldsymbol{c}), \tag{2.31}$$

where

$$\begin{aligned}
\widehat{\mathcal{Z}}^{(1)}_{\boldsymbol{p}}&=\sum_{\boldsymbol{\mu}\in\Lambda^\star/\Lambda}\widehat{h}_{\boldsymbol{p},\boldsymbol{\mu}}\,\theta_{\boldsymbol{p},\boldsymbol{\mu}},\\
\widehat{\mathcal{Z}}^{(2)}_{\boldsymbol{p}}&=\frac{1}{2}\sum_{\boldsymbol{p}_1+\boldsymbol{p}_2=\boldsymbol{p}}\ \sum_{\boldsymbol{\mu}_i\in\Lambda^\star/\Lambda_i}\widehat{h}_{\boldsymbol{p}_1,\boldsymbol{\mu}_1}\,\widehat{h}_{\boldsymbol{p}_2,\boldsymbol{\mu}_2}\,\widehat{\Psi}_{\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}.
\end{aligned} \tag{2.32}$$

Since the modular anomaly of $h_{\boldsymbol{p},\boldsymbol{\mu}}$ is expected to arise when the divisor $\mathcal{D}$ can split into several components, we expect that $R_{\boldsymbol{p},\boldsymbol{\mu}}$ should be controlled by the product of the corresponding MSW invariants. At this point, however, the function $R_{\boldsymbol{p},\boldsymbol{\mu}}$ remains still undetermined. We shall now fix it by comparing the above construction with the analysis of the D3-instantons corrections to the hypermultiplet metric.

2.3 Comparison with the contact potential

In our study of instanton effects on the hypermultiplet moduli space $\mathcal{M}_H$ in the twistor formalism, we shall find in section 4 that D3-instanton contributions to the contact potential in the two-instanton approximation can be expressed in terms of the following function:

$$\sum_{\boldsymbol{\mu}\in\Lambda^\star/\Lambda} h_{\boldsymbol{p},\boldsymbol{\mu}}\,\theta_{\boldsymbol{p},\boldsymbol{\mu}} + \frac{1}{2}\sum_{\boldsymbol{p}_1+\boldsymbol{p}_2=\boldsymbol{p}}\ \sum_{\boldsymbol{\mu}_i\in\Lambda^\star/\Lambda_i} h_{\boldsymbol{p}_1,\boldsymbol{\mu}_1}\, h_{\boldsymbol{p}_2,\boldsymbol{\mu}_2}\left(\Psi_{\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2} + \Psi^{(+)}_{\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\right). \tag{2.33}$$

In order for the metric on $\mathcal{M}_H$ to carry an isometric action of the modular group, it is necessary that (2.33) be a modular form of weight $(-\frac{3}{2},\frac{1}{2})$.

On the other hand, the completed BPS partition function (2.31) differs from (2.33) in two ways: the modular forms $\widehat{h}$ are replaced by their non-completed version h, and in the second term the contribution of $\Psi^{(-)}$ is missing. Remarkably, these two differences cancel amongst each other provided

$$\sum_{\boldsymbol{\mu}\in\Lambda^\star/\Lambda} R_{\boldsymbol{p},\boldsymbol{\mu}}\,\theta_{\boldsymbol{p},\boldsymbol{\mu}} = \sum_{\boldsymbol{p}_1+\boldsymbol{p}_2=\boldsymbol{p}}\ \sum_{\boldsymbol{\mu}_i\in\Lambda^\star/\Lambda_i} h_{\boldsymbol{p}_1,\boldsymbol{\mu}_1}\, h_{\boldsymbol{p}_2,\boldsymbol{\mu}_2}\,\Psi^{(-)}_{\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}. \tag{2.34}$$

In more detail, this condition ensures that the complementary terms, appearing due to the completion of h to $\widehat{h}$ in $\widehat{\mathcal{Z}}^{(1)}_{\boldsymbol{p}}$, cancel a part of the additional terms in $\widehat{\mathcal{Z}}^{(2)}_{\boldsymbol{p}}$, while the remaining discrepancy due to the difference between h and $\widehat{h}$ in $\widehat{\mathcal{Z}}^{(2)}_{\boldsymbol{p}}$ is of higher order in the expansion in MSW invariants. In Appendix B, we show that the condition (2.34) is solved by choosing $R_{\boldsymbol{p},\boldsymbol{\mu}}$ as in (1.3), where the functions $\mathrm{S}_{\boldsymbol{p}_1,\boldsymbol{p}_2}$, $\mathrm{Q}_{\boldsymbol{p}_1,\boldsymbol{p}_2}$ and the variables $\tilde{\boldsymbol{\mu}}$, $\boldsymbol{\nu}_i$ are defined in (B.11), (B.13), (B.8) and (B.14), respectively. This shows that in order for the contact potential to have the right modular property, the generating function $h_{\boldsymbol{p},\boldsymbol{\mu}}$ of MSW invariants must have an anomalous modular transformation. Its modular completion is provided by the non-holomorphic function (1.3), constructed out of the generating functions $h_{\boldsymbol{p}_1,\boldsymbol{\mu}_1}$ and $h_{\boldsymbol{p}_2,\boldsymbol{\mu}_2}$ of MSW invariants associated to all possible decompositions $\boldsymbol{p}=\boldsymbol{p}_1+\boldsymbol{p}_2$. In the next subsection we demonstrate that $h_{\boldsymbol{p},\boldsymbol{\mu}}$ is actually a vector-valued mixed mock modular form.

It is worth stressing that the result above is valid if $\mathcal{D}$ can be written as a sum $\mathcal{D}=\mathcal{D}_1+\mathcal{D}_2$ for *at most* two effective divisors $\mathcal{D}_1$ and $\mathcal{D}_2$. In particular, $h_{\boldsymbol{p}_i,\boldsymbol{\mu}_i}$ are modular forms, since $\boldsymbol{p}_i$ cannot be further decomposed as a sum of effective charges $\boldsymbol{p}_i=\boldsymbol{p}_{i,1}+\boldsymbol{p}_{i,2}$. If $\boldsymbol{p}$ can be written as a sum of more than two effective $\boldsymbol{p}_i$, then (1.3) will involve further corrections of higher order in MSW invariants. It is reassuring to note that (1.3) is consistent with explicit expressions which are available for various non-compact Calabi-Yau's, given by canonical bundles over a rational surface S. For instance, setting $\boldsymbol{p}_1=\boldsymbol{p}_2$ in (1.3), it reproduces the result of [21, Eq. (4.30)] and [22, Section 3.2] for $\boldsymbol{t}=-K_S$, where K_S is the canonical class of the surface S.

2.4 Mock modularity and the MSW elliptic genus

Having deduced the modular completion of the generating function of MSW invariants $h_{\boldsymbol{p},\boldsymbol{\mu}}$, which appears as a building block of the contact potential, we shall now compare its properties with mock modular forms and consider its implications for the elliptic genus of the MSW conformal field theory.

First we recall a few relevant aspects of mock modular forms [20, 29]. Let $g(\tau)$ be a holomorphic modular form of weight $2-k$. The "shadow map" maps g to the non-holomorphic function g^* defined by

$$g^*(\tau) = (\mathrm{i}/2)^{k-1} \int_{-\bar{\tau}}^{\infty} (z+\tau)^{-k}\, \overline{g(-\bar{z})}\, dz. \tag{2.35}$$

A mock modular form of weight k and with shadow g, is a holomorphic function $h(\tau)$ such that its non-holomorphic completion

$$\widehat{h} = h + g^* \tag{2.36}$$

transforms as a modular form of weight k. Acting with the shadow operator $\tau_2^2\partial_{\bar{\tau}}$ on $\widehat{h}$ gives

$$\tau_2^2\partial_{\bar{\tau}}\widehat{h} = \tau_2^{2-k}\, \overline{g}\,, \tag{2.37}$$

from which the shadow g is easily obtained by multiplication with τ_2^{k-2} and complex conjugation. Note that the r.h.s. transforms as a modular form of weight $k-2$.

More generally, a mixed mock modular form of weight k [29] is a holomorphic function $h(\tau)$ such that there exists (half) integer numbers r_j and modular forms f_j and g_j, respectively with weights $k+r_j$ and $2+r_j$, such that the completion

$$\widehat{h} = h + \sum_j f_j\, g_j^* \tag{2.38}$$

transforms as a modular form of weight k. Acting with the shadow operator on $\widehat{h}$ one obtains

$$\tau_2^2\partial_{\bar{\tau}}\widehat{h} = \sum_j \tau_2^{2+r_j} f_j\, \overline{g}_j. \tag{2.39}$$

Let us now return to the function $h_{\boldsymbol{p},\boldsymbol{\mu}}$ and its completion $\widehat{h}_{\boldsymbol{p},\boldsymbol{\mu}}$ (2.30). Applying the shadow operator $\tau_2^2\partial_{\bar{\tau}}$ to $\widehat{h}_{\boldsymbol{p},\boldsymbol{\mu}}$, one finds

$$\begin{aligned}\tau_2^2\partial_{\bar{\tau}}\, \widehat{h}_{\boldsymbol{p},\boldsymbol{\mu}}(\tau) = \frac{\sqrt{2\tau_2}}{32\pi\mathrm{i}} &\sum_{\boldsymbol{p}_1+\boldsymbol{p}_2=\boldsymbol{p}}\ \sum_{\boldsymbol{\mu}_i\in\Lambda^\star/\Lambda_i} h_{\boldsymbol{p}_1,\boldsymbol{\mu}_1}(\tau)\, h_{\boldsymbol{p}_2,\boldsymbol{\mu}_2}(\tau)\, \sqrt{(pp_1p_2)} \\ &\times \sum_{\boldsymbol{\rho}\in(\Lambda_1-\tilde{\mu})\cap(\Lambda_2+\tilde{\mu})} (-1)^{\mathrm{S}_{\boldsymbol{p}_1,\boldsymbol{p}_2}(\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\boldsymbol{\rho})}\, e^{-2\pi\tau_2\frac{(\mathrm{S}_{\boldsymbol{p}_1,\boldsymbol{p}_2}(\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\boldsymbol{\rho}))^2}{(pp_1p_2)}+\pi\mathrm{i}\tau\mathrm{Q}_{\boldsymbol{p}_1,\boldsymbol{p}_2}(\boldsymbol{\nu}_1,\boldsymbol{\nu}_2)}\,,\end{aligned} \tag{2.40}$$

where the various symbols are defined in Appendix B. To see that the modular weights match on the two sides of the equation, we observe from (B.13) and (B.14) that the sum over $\boldsymbol{\rho}$ runs over a lattice with signature $(b_2-1,1)$, therefore the second line of (2.40) is a theta series of weight $\frac{1}{2}(b_2-1,1)$. Combining this with the weights of $\sqrt{\tau_2}$ and $h_{\boldsymbol{p}_i,\boldsymbol{\mu}_i}$, the total weight of the right hand side evaluates to $-(\frac{1}{2}b_2+3,0)$, consistently with the left-hand side.

Furthermore, the theta series on the second line of (2.40) can be expressed as a sum of holomorphic theta series of weight $\frac{1}{2}(b_2-1)$ times anti-holomorphic theta series of weight $\frac{1}{2}$, such that (2.40) can be brought to the form (2.38). This shows that $h_{\boldsymbol{p},\boldsymbol{\mu}}$ is a (vector-valued) mixed mock modular form with $r_j=-\frac{3}{2}$ for all j.

To get more insight into this anomaly, we now consider the completed elliptic genus, defined by the first term in the completed BPS partition function (2.31),

$$\widehat{\chi}_{\boldsymbol{p}}(\tau,\boldsymbol{z},\boldsymbol{c}) \equiv \widehat{\mathcal{Z}}^{(1)}_{\boldsymbol{p}} = \sum_{\boldsymbol{\mu}\in\Lambda^*/\Lambda} \widehat{h}_{\boldsymbol{p},\boldsymbol{\mu}}(\tau)\,\theta_{\boldsymbol{p},\boldsymbol{\mu}}(\tau,\boldsymbol{t},\boldsymbol{b},\boldsymbol{c}). \tag{2.41}$$

The analogue of the shadow operator for $\widehat{\chi}_{\boldsymbol{p}}$ is

$$\overline{\mathcal{D}} = \tau_2^2\left(\partial_{\bar\tau} - \frac{\mathrm{i}}{4\pi}\,\partial^2_{c_+} + \frac{b_+}{2}\,\partial_{c_+} + \frac{\pi\mathrm{i}}{4}\,b_+^2\right), \tag{2.42}$$

where $+$ indicates the projection to $\boldsymbol{t}$. This differential operator annihilates $\theta_{\boldsymbol{p},\boldsymbol{\mu}}$ in (2.41), such that its action on $\widehat{\chi}_{\boldsymbol{p}}$ is only non-vanishing if $\widehat{h}_{\boldsymbol{p},\boldsymbol{\mu}}$ is non-holomorphic or, equivalently, $\boldsymbol{p}$ is reducible.

We shall now show that in the special case where $\boldsymbol{p}=2\boldsymbol{p}_0$ with $\boldsymbol{p}_0$ irreducible, the mock modularity of $h_{\boldsymbol{p},\boldsymbol{\mu}}$ is such that the "holomorphic anomaly" $\overline{\mathcal{D}}\,\widehat{\chi}_{2\boldsymbol{p}_0}$ is proportional to $(\chi_{\boldsymbol{p}_0})^2$. Note that since we restrict to $\boldsymbol{p}$ which can be written as a sum of at most two effective divisors, this implies that $\boldsymbol{p}_1=\boldsymbol{p}_2=\boldsymbol{p}_0$. We furthermore specialize the (real) Kähler modulus $\boldsymbol{t}$ to the large volume attractor point $\lim_{\lambda\to\infty}\lambda\,\boldsymbol{p}_0$ (2.10), which is the attractor point of $\boldsymbol{t}$ at the horizon geometry $\mathrm{AdS}_3\times S^2\times\mathfrak{Y}$ of the M5-brane [46]. In $\widehat{\chi}_{\boldsymbol{p}}$ the magnitude of $\boldsymbol{t}$ is actually irrelevant, and we could just as well set $\boldsymbol{t}=\boldsymbol{p}_0$. The scale invariance also implies that the attractor points for the MSW field theories corresponding to magnetic charges $\boldsymbol{p}_0$ and $2\boldsymbol{p}_0$ are equal.

Using these specializations in (2.34) and the expressions (2.26) and (2.28), we find that the completion of $\widehat{\chi}_{2\boldsymbol{p}_0}$ is obtained by adding to $\chi_{2\boldsymbol{p}_0}$ the following term

$$\begin{aligned}&\frac{1}{8\pi}\sum_{\boldsymbol{\mu}_i\in\Lambda^*/\Lambda_i} h_{\boldsymbol{p}_0,\boldsymbol{\mu}_1}(\tau)\,h_{\boldsymbol{p}_0,\boldsymbol{\mu}_2}(\tau)\sum_{\boldsymbol{k}_i\in\Lambda_i+\boldsymbol{\mu}_i+\frac12\boldsymbol{p}_0}(-1)^{\boldsymbol{p}_0\cdot(\boldsymbol{k}_1+\boldsymbol{k}_2)+p_0^3}|\boldsymbol{p}_0\cdot(\boldsymbol{k}_1-\boldsymbol{k}_2)|\\ &\times\beta_{\frac32}\!\left(\frac{\tau_2}{p_0^3}\,(\boldsymbol{p}_0\cdot(\boldsymbol{k}_1-\boldsymbol{k}_2))^2\right)e^{\frac{\pi\tau_2}{p_0^3}(\boldsymbol{p}_0\cdot(\boldsymbol{k}_1-\boldsymbol{k}_2))^2}\,\mathcal{X}^{(\theta)}_{\boldsymbol{p}_0,\boldsymbol{k}_1}\mathcal{X}^{(\theta)}_{\boldsymbol{p}_0,\boldsymbol{k}_2}.\end{aligned} \tag{2.43}$$

Computing the action of $\overline{\mathcal{D}}$ on this term, we obtain that the holomorphic anomaly of $\widehat{\chi}_{2\boldsymbol{p}_0}$ is proportional to the square of $\chi_{\boldsymbol{p}_0}$,

$$\overline{\mathcal{D}}\,\widehat{\chi}_{2\boldsymbol{p}_0} = (-1)^{p_0^3}\,\frac{\sqrt{\tau_2 p_0^3}}{16\pi\mathrm{i}}\,\chi^2_{\boldsymbol{p}_0}. \tag{2.44}$$

This extends the holomorphic anomaly of $\mathcal{N}=4$ $U(2)$ gauge theory on (local) surfaces [21, 22] to divisors in compact threefolds of the form $2\boldsymbol{p}_0$ with $\boldsymbol{p}_0$ irreducible. When the divisor $\boldsymbol{p}=\boldsymbol{p}_1+\boldsymbol{p}_2$ is primitive, and therefore $\boldsymbol{p}_1\neq\boldsymbol{p}_2$, the shadow does not seem to take such a factorized form. A possible explanation is that, whereas for $\boldsymbol{p}_1=\boldsymbol{p}_2$ the large volume attractor points agree for magnetic charges $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ and $\boldsymbol{p}_1+\boldsymbol{p}_2$, this is not the case if $\boldsymbol{p}_1\neq\boldsymbol{p}_2$.

3. D3-instantons in the twistor formalism

In this section, we briefly review the twistorial construction of the D-instanton corrected metric on the hypermultiplet space $\mathcal{M}_H$, with emphasis on D3-instanton corrections in the large volume limit. More details can be found in the reviews [1, 2] and in the original works [8, 5, 10, 12, 47].

3.1 Hypermultiplet moduli space in type IIB string theory on a CY threefold

The hypermultiplet moduli space $\mathcal{M}_H$ is a quaternion-Kähler manifold of dimension $4b_2+4$, which describes the dynamics of the ten-dimensional axio-dilaton $\tau = c^0 + \mathrm{i}/g_s$, the Kähler moduli $z^a = b^a + \mathrm{i}t^a$, the Ramond-Ramond (RR) scalars $c^a, \tilde{c}_a, \tilde{c}_0$, corresponding to periods of the RR 2-form, 4-form and 6-form on a basis of $H^{\rm even}(\mathfrak{Y}, \mathbb{Z})$, and finally, the NS axion ψ, dual to the Kalb-Ramond two-form B in four dimensions. At tree-level, the metric on $\mathcal{M}_H$ is obtained from the moduli space $\mathcal{M}_{\mathcal{SK}}$ of complexified Kähler deformations via the c-map construction [48, 49]. The special Kähler manifold $\mathcal{M}_{\mathcal{SK}}$ is characterized by the holomorphic prepotential $F(X)$ where X^Λ are homogeneous complex coordinates on $\mathcal{M}_{\mathcal{SK}}$ such that $X^\Lambda/X^0 = z^\Lambda$ (with $z^0 = 1$). Classically, the prepotential is determined by the intersection numbers $F^{\rm cl}(X) = -\kappa_{abc}\frac{X^a X^b X^c}{6X^0}$, whereas quantum mechanically it is affected by α'-corrections, which are however suppressed in the large volume limit.

As mentioned in the introduction, beyond the tree-level the metric on $\mathcal{M}_H$ receives quantum g_s-corrections. At the perturbative level there is only a one-loop correction, proportional to the Euler characteristics $\chi_{\mathfrak{Y}}$. The corresponding metric is a one-parameter deformation of the c-map found explicitly in a series of works [50, 51, 52, 53, 54]. At the non-perturbative level, there are corrections from D-branes wrapped on complex cycles in $\mathfrak{Y}$ (described by coherent sheaves on $\mathfrak{Y}$), and from NS5-branes wrapped on $\mathfrak{Y}$, which we ignore in this paper.

Before recalling how D-brane instantons affect the metric, a few words are in order about the symmetries of $\mathcal{M}_H$. In the classical, large volume limit, the metric is invariant under the semi-direct product of $SL(2,\mathbb{R})$ times the graded nilpotent algebra $N = N^{(1)} \oplus N^{(2)} \oplus N^{(3)}$, where the generators in $N^{(1)}, N^{(2)}, N^{(3)}$ transform as b_2 doublets, b_2 singlets and one doublet under $SL(2,\mathbb{R})$, respectively. Quantum corrections break this continuous symmetry, but are expected to preserve an isometric action of the discrete subgroup $SL(2,\mathbb{Z}) \ltimes N(\mathbb{Z})$, where $SL(2,\mathbb{Z})$ descends from S-duality group of type IIB supergravity, while the nilpotent factor corresponds to monodromies around the large volume point and large gauge transformations of the RR and Kalb-Ramond fields. Under an element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, the type IIB fields transforms as in (2.25), supplemented by the following action on $\tilde{c}_a, \tilde{c}_0, \psi$,

$$\tilde{c}_a \mapsto \tilde{c}_a - c_{2,a}\varepsilon(g) \ , \qquad \begin{pmatrix} \tilde{c}_0 \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \tilde{c}_0 \\ \psi \end{pmatrix} \ , \tag{3.1}$$

where $\varepsilon(g)$ is the logarithm of the multiplier system of the Dedekind eta function [38, 12]. In the absence of D5 and NS5-brane instantons, the metric admits two additional continuous isometries, acting by shifts of $\tilde{c}_0$ and ψ.

3.2 Twistorial construction of D-instantons

Quantum corrections to $\mathcal{M}_H$ are most easily described using the language of twistors. The twistor space $\mathcal{Z}$ of $\mathcal{M}_H$ is a $\mathbb{C}P^1$-bundle over $\mathcal{M}_H$ endowed with a complex contact structure. This contact structure is represented by a (twisted) holomorphic one-form $\mathcal{X}$, which locally can always be expressed in terms of complex Darboux coordinates as

$$\mathcal{X}^{[i]} = \mathrm{d}\alpha^{[i]} + \tilde{\xi}_\Lambda^{[i]} \mathrm{d}\xi^\Lambda_{[i]} \,, \tag{3.2}$$

where the index $[i]$ labels the patches $\mathcal{U}_i$ of an open covering of $\mathbb{C}P^1$. The global contact structure on $\mathcal{Z}$ (hence, the metric on $\mathcal{M}$) is then encoded in contact transformations between Darboux coordinate systems on the overlaps $\mathcal{U}_i \cap \mathcal{U}_j$. It is convenient to parametrize these transformations by holomorphic functions $H^{[ij]}(\xi^\Lambda, \tilde\xi_\Lambda, \alpha)$, known as contact hamiltonians[6] [55], which generate the contact transformations by exponentiating the action of the vector field

$$X_H = \left(-\partial_{\tilde\xi_\Lambda} H + \xi^\Lambda \partial_\alpha H\right) \partial_{\xi^\Lambda} + \partial_{\xi^\Lambda} H \, \partial_{\tilde\xi_\Lambda} + \left(H - \xi^\Lambda \partial_{\xi^\Lambda} H\right) \partial_\alpha. \tag{3.3}$$

Thus, a set of such holomorphic functions associated to a covering of $\mathbb{C}P^1$ (satisfying obvious consistency conditions on triple overlaps) uniquely defines a quaternion-Kähler manifold.

To extract the metric from these data, the first step is to express the Darboux coordinates in terms of coordinates on $\mathcal{M}_H$ and the stereographic coordinate t on $\mathbb{C}P^1$. They are fixed by regularity properties and the gluing conditions

$$(\xi^\Lambda_{[j]}, \tilde\xi^{[j]}_\Lambda, \alpha^{[j]}) = e^{X_{H^{[ij]}}} \cdot (\xi^\Lambda_{[i]}, \tilde\xi^{[i]}_\Lambda, \alpha^{[i]}), \tag{3.4}$$

which typically can be rewritten as a system of integral equations. Once the Darboux coordinates are found, it is sufficient to plug them into the contact one-form (3.2), expand around any point $t \in \mathbb{C}P^1$ and read off the components of the $SU(2)$ part of the Levi-Civita connection $\vec{p}$. E.g. around the point $t = 0$, the expansion reads

$$\mathcal{X}^{[i]} = -4\mathrm{i}\, e^{\Phi^{[i]}} \left(\frac{\mathrm{d}t}{t} + \frac{p_+}{t} - \mathrm{i}\, p_3 + p_- t\right). \tag{3.5}$$

The scale factor $e^{\Phi^{[i]}}$ is known as the contact potential [8]. In the case when $\mathcal{M}_H$ has a continuous isometry and the Darboux coordinates are chosen such that this isometry lifts to the vector field ∂_α, Φ is globally well-defined (i.e. independent of the patch index $[i]$) and is independent of the fiber coordinate t. Thus, it becomes a function on $\mathcal{M}_H$ which, in fact, coincides with the norm of the moment map associated to the isometry [9].

In this formalism the D-instanton corrected hypermultiplet moduli space, with NS5-brane instantons being ignored, was constructed in [5, 10]. We omit the details of this construction and present only those elements which are relevant for the analysis of the contact potential.

- The contact hamiltonians enforcing D-instanton corrections to the metric are given by

$$H_\gamma(\xi, \tilde\xi) = \frac{\bar\Omega(\gamma)}{(2\pi)^2}\, \sigma_\gamma \mathcal{X}_\gamma\,, \tag{3.6}$$

  where $\mathcal{X}_\gamma = \mathbf{E}\left(p^\Lambda \tilde\xi_\Lambda - q_\Lambda \xi^\Lambda\right)$, $\bar\Omega(\gamma)$ are rational DT invariants (2.9), and σ_γ is a quadratic refinement of the intersection pairing on $H^{\mathrm{even}}(\mathfrak{Y})$, a sign factor which we fix below. They generate contact transformations connecting Darboux coordinates on the two sides of the BPS rays ℓ_γ on $\mathbb{C}P^1$ extending from $t = 0$ to $t = \infty$, along the direction fixed by the central charge

$$\ell_\gamma = \{t \in \mathbb{C}P^1 \ : \ Z_\gamma / t \in \mathrm{i}\mathbb{R}^-\}. \tag{3.7}$$

[6] The contact hamiltonians coincide with the generating functions introduced in [8] in the special case where $H^{[ij]}$ is independent of $\tilde\xi_\Lambda$ and α.

- The Darboux coordinates are obtained by solving the following integral equations

$$\mathcal{X}_\gamma(t) = \mathcal{X}^{\rm sf}_\gamma(t)\, \mathbf{E}\left(\frac{1}{8\pi^2}\sum_{\gamma'}\sigma_{\gamma'}\bar\Omega(\gamma')\,\langle\gamma,\gamma'\rangle\int_{\ell_{\gamma'}}\frac{\mathrm{d}t'}{t'}\,\frac{t+t'}{t-t'}\,\mathcal{X}_{\gamma'}(t')\right), \tag{3.8}$$

where

$$\mathcal{X}^{\rm sf}_\gamma(t) = \mathbf{E}\left(\frac{\tau_2}{2}\left(\bar Z_\gamma(\bar u)\, t - \frac{Z_\gamma(u)}{t}\right) + p^\Lambda\tilde\zeta_\Lambda - q_\Lambda\zeta^\Lambda\right) \tag{3.9}$$

are the Fourier modes of the tree-level (or 'semi-flat') Darboux coordinates valid in the absence of D-instantons.[7] In the weak coupling limit, these equations can be solved iteratively, leading to a (formal) multi-instanton series. This gives ξ^Λ and $\tilde\xi_\Lambda$ in each angular sector, which can then be used to compute the Darboux coordinate α, whose explicit expression can be found in [10] and will not be needed in this paper. These equations provide Darboux coordinates as functions of the fiber coordinate t and variables $(\tau_2, u^a, \zeta^\Lambda, \tilde\zeta_\Lambda, \sigma)$ which play the role of coordinates on $\mathcal{M}_H$. They are adapted to the symmetries of the type IIA formulation and therefore can be considered as natural coordinates on the moduli space of type IIA string theory compactified on the mirror Calabi-Yau threefold. Their relation to the type IIB fields will be explained in the next subsection.

- Given the Darboux coordinates $\mathcal{X}_\gamma$, the contact potential e^Φ is obtained from the Penrose-type integrals

$$e^\Phi = \frac{\mathrm{i}\tau_2^2}{16}\left(\bar u^\Lambda F_\Lambda - u^\Lambda \bar F_\Lambda\right) - \frac{\chi_{\mathfrak{Y}}}{192\pi} + \frac{\mathrm{i}\tau_2}{64\pi^2}\sum_\gamma \sigma_\gamma\Omega(\gamma)\int_{\ell_\gamma}\frac{\mathrm{d}t}{t}\left(t^{-1}Z_\gamma(u) - t\bar Z_\gamma(\bar u)\right)\mathcal{X}_\gamma. \tag{3.10}$$

3.3 S-duality and mirror map

The D3-instanton corrected metric is obtained from the construction above by assuming that the only non-vanishing DT invariants $\Omega(p^\Lambda, q_\Lambda; z^a)$ are those where the D5-brane charge p^0 vanishes. While we expect that this metric should carry an isometric action of $SL(2,\mathbb{Z})$, this symmetry is far from being manifest. Indeed, the construction above is adapted to symplectic invariance, which is manifest in type IIA formulation, rather than to S-duality, which is explicit on the type IIB side. In particular, the Darboux coordinates are defined by (3.8) in terms of type IIA variables. In order to understand their behavior under S-duality, they should be rewritten instead in terms of type IIB variables, which we *define* by their transformation properties (2.25), (3.1). We refer to the change of coordinates from IIA to IIB variables as the *mirror map*.

In the classical approximation (i.e. tree-level, large volume limit), the mirror map was found in [56] and is given by

$$\begin{aligned}
& u^a = b^a + \mathrm{i}t^a\,, \qquad \zeta^0 = \tau_1\,, \\
& \tilde\zeta_a = \tilde c_a + \frac{1}{2}\,\kappa_{abc}\, b^b(c^c - \tau_1 b^c)\,, \qquad \tilde\zeta_0 = \tilde c_0 - \frac{1}{6}\,\kappa_{abc}\, b^a b^b(c^c - \tau_1 b^c)\,, \\
& \sigma = -(2\psi + \tau_1\tilde c_0) + \tilde c_a(c^a - \tau_1 b^a) - \frac{1}{6}\,\kappa_{abc}\, b^a c^b(c^c - \tau_1 b^c)\,.
\end{aligned} \tag{3.11}$$

[7] The argument u^a of the central charge Z_γ can be understood as the complex structure moduli in the mirror threefold and will be related to the Kähler moduli z^a of the threefold $\mathfrak{Y}$ by the mirror map.

One can check that, if one substitutes these expressions into the classical Darboux coordinates, obtained by dropping all integrals and retaining only the classical part the holomorphic prepotential $F(X)$, and supplement the $SL(2,\mathbb{Z})$ transformations of the type IIB fields by the following fractional transformation of the fiber coordinate

$$t \mapsto \frac{c\tau_2 + t(c\tau_1 + d) + t|c\tau + d|}{(c\tau_1 + d) + |c\tau + d| - tc\tau_2}\,, \tag{3.12}$$

the resulting Darboux coordinates transform holomorphically as in [12, Eq.(2.20)]. Moreover, it is straightforward to check that this transformation preserves the contact structure since it rescales the contact one-form by a holomorphic factor,

$$\mathcal{X} \mapsto \frac{\mathcal{X}}{c\xi^0 + d}. \tag{3.13}$$

This demonstrates that $SL(2,\mathbb{Z})$ acts isometrically on $\mathcal{M}_H$ in the classical approximation.

To go beyond this approximation, one must ensure that, even after inclusion of quantum corrections into the Darboux coordinates, $SL(2,\mathbb{Z})$ still acts on them by a holomorphic contact transformation. The main complication comes from the fact that the mirror map itself gets corrected. Thus, the key problem is to find corrections to (3.11) such that the resulting Darboux coordinates transform holomorphically and the contact one-form satisfies (3.13).

For the pure D1-D(-1)-instantons this problem was solved in [57]. Furthermore, it was shown that after a local contact transformation, the instanton corrected Darboux coordinates transform exactly as the classical ones, and the description of the twistor space in terms of a covering of $\mathbb{C}P^1$ and contact hamiltonians takes a manifestly modular invariant form. Then in [58] it was understood how to derive the mirror map for generic QK manifolds obtained by a deformation of the c-map and preserving two-continuous isometries, of which $\mathcal{M}_H$ corrected by D3-instantons is a particular case.[8] The idea is that the mirror map is induced by converting the kernel $\frac{\mathrm{d}t'}{t'}\,\frac{t'+t}{t'-t}$, appearing in the integral equations for Darboux coordinates and transforming non-trivially under S-duality, into a modular invariant kernel. In [12] these results were applied to D3-instantons in the one-instanton, large volume approximation. We summarize them in the next subsection. But before that we make two comments.

First, it is convenient to redefine the coordinate t by a Cayley transformation:

$$z = \frac{t + \mathrm{i}}{t - \mathrm{i}}\,. \tag{3.14}$$

The transformation (3.12), lifting the $SL(2,\mathbb{Z})$ action on $\mathcal{M}_H$ to a holomorphic contact transformation in twistor space, then takes the simpler form

$$z \mapsto \frac{c\bar\tau + d}{|c\tau + d|}\, z\,. \tag{3.15}$$

In particular, the two points $t = \mp\mathrm{i}$ on $\mathbb{C}P^1$, which stay invariant under the $SL(2,\mathbb{Z})$ action, are mapped to $z = 0$ and $z = \infty$, which makes it easier to do a Fourier expansion around them. Secondly, the fact that $\frac{\mathrm{d}t}{t}$ transforms into $\frac{|c\tau+d|}{c\xi^0+d}\,\frac{\mathrm{d}t}{t}$ under (3.12), along with (3.5) and

[8] In this class of geometries the action of $SL(2,\mathbb{Z})$ on the fiber coordinate (3.12) remains uncorrected, unlike in the case without any continuous isometries considered in [59].

(3.13), shows that the contact potential e^Φ must transform as a modular form of weight $(-\frac{1}{2},-\frac{1}{2})$ [5],

$$e^\Phi \mapsto \frac{e^\Phi}{|c\tau+d|}\,. \tag{3.16}$$

3.4 D3-instantons in the one-instanton approximation: a summary

Here we summarize the results obtained in [12] which are relevant for the evaluation of the contact potential in the next section. First, we recall that they are derived in the one-instanton, large volume approximation, which means that one restricts to the first order in the expansion in the DT or MSW invariants and takes the limit $t^a\to\infty$. In this limit, the integrals along BPS rays ℓ_γ are dominated by a saddle point at

$$z'_\gamma \approx -\mathrm{i}\frac{(q+b)_+}{\sqrt{(pt^2)}} \tag{3.17}$$

for $(pt^2)>0$ and $z'_{-\gamma}=1/z'_\gamma$ in the opposite case. This shows that in all integrands we can send z' either to zero or infinity keeping constant $t^a z'$ or t^a/z', respectively.

Next, we should fix the quadratic refinement appearing in (3.6). We choose it to be $\sigma_\gamma=\mathbf{E}\big(\frac{1}{2}\,p^a q_a\big)\,\sigma_{\boldsymbol{p}}$ where $\sigma_{\boldsymbol{p}}=\mathbf{E}\big(\frac{1}{2}\,A_{ab}p^a p^b\big)$ and A_{ab} is the matrix satisfying

$$A_{ab}p^p-\frac{1}{2}\,\kappa_{abc}p^b p^c\in\mathbb{Z}\quad \text{for } \forall p^a\in\mathbb{Z}, \tag{3.18}$$

and performing the symplectic rotation making the charges integer valued (see footnote 5). It is easy to check that such quadratic refinement satisfies the defining relation

$$\sigma_{\gamma_1}\sigma_{\gamma_2}=(-1)^{\langle\gamma_1,\gamma_2\rangle}\sigma_{\gamma_1+\gamma_2}. \tag{3.19}$$

With these definitions one has the following results:

- *Quantum mirror map.* The following expression

$$u^a=b^a+\mathrm{i}t^a-\frac{\mathrm{i}}{8\pi^2\tau_2}\sum_{\gamma\in\Gamma_+}\sigma_\gamma\bar\Omega(\gamma)p^a\left[\int_{\ell_\gamma}\mathrm{d}z\,(1-z)\,\mathcal{X}_\gamma+\int_{\ell_{-\gamma}}\frac{\mathrm{d}z}{z^3}(1-z)\,\mathcal{X}_{-\gamma}\right] \tag{3.20}$$

  replaces the simple relation (3.11) between the complex structure moduli and the type IIB fields. The other relations of the classical mirror map also get corrections due to D3-instantons, but are not needed for the purposes of this paper.

- *Darboux coordinates.* If one defines the instanton expansion of the Fourier modes $\mathcal{X}_\gamma$ as

$$\mathcal{X}_\gamma=\mathcal{X}^{\mathrm{cl}}_\gamma\left(1+\mathcal{X}^{(1)}_\gamma+\cdots\right), \tag{3.21}$$

  then for $\gamma\in\Gamma_+$ one finds[9]

$$\mathcal{X}^{\mathrm{cl}}_\gamma = e^{-2\pi S^{\mathrm{cl}}_{\boldsymbol{p}}}\,\mathbf{E}(-\hat q_0\tau+\mathrm{i}Q_\gamma(z))\,\mathcal{X}^{(\theta)}_{\boldsymbol{p},\boldsymbol{q}}, \tag{3.22}$$

$$\mathcal{X}^{(1)}_\gamma = \frac{1}{2\pi}\sum_{\gamma'\in\Gamma_+}\sigma_{\gamma'}\Omega(\gamma')\int_{\ell_{\gamma'}}\mathrm{d}z'\left((tpp')-\frac{\mathrm{i}\langle\gamma,\gamma'\rangle}{z'-z}\right)\mathcal{X}^{\mathrm{cl}}_{\gamma'}, \tag{3.23}$$

[9] Although (3.23) did not appear explicitly in [12], it can be easily obtained from Eqs. (4.5) and (4.7) of that paper.

where $\mathcal{X}^{(\theta)}_{\boldsymbol{p},\boldsymbol{q}}$ was defined in (2.23) and gives rise to the usual Siegel theta series, $S^{\rm cl}_{\boldsymbol{p}}$ is the leading part of the Euclidean D3-brane action in the large volume limit, and $Q_\gamma(z)$ is the only part of $\mathcal{X}^{\rm cl}_\gamma$ depending on z,

$$S^{\rm cl}_{\boldsymbol{p}} = \frac{\tau_2}{2}\,(pt^2) - \mathrm{i}p^a\tilde c_a, \qquad Q_\gamma(z) = \tau_2(pt^2)\left(z + \mathrm{i}\,\frac{(q+b)_+}{\sqrt{(pt^2)}}\right)^2. \tag{3.24}$$

Note that $\mathcal{X}^{\rm cl}_\gamma$ is the part of $\mathcal{X}^{\rm sf}_\gamma$ (3.9) obtained by using the classical mirror map (3.11), whereas $\mathcal{X}^{(1)}_\gamma$ has two contributions: one from the integral term in the equation (3.8) and another from quantum corrections to the mirror map. For the opposite charge, the results can be obtained via the complex conjugation and the antipodal map, $\mathcal{X}_{-\gamma}(z) = \overline{\mathcal{X}_\gamma(-\bar z^{-1})}$.

- *Contact potential.* The D3 one-instanton contribution to the contact potential takes the simple form

$$\delta^{(1)} e^\Phi = \frac{\tau_2}{2}\,\mathrm{Re}\sum_{\boldsymbol{p}} \mathcal{D}_{-\frac{3}{2}}\mathcal{F}^{(1)}_{\boldsymbol{p}}, \tag{3.25}$$

where

$$\mathcal{D}_{\mathfrak{h}} = \frac{1}{2\pi\mathrm{i}}\left(\partial_\tau + \frac{\mathfrak{h}}{2\mathrm{i}\tau_2} + \frac{\mathrm{i}t^a}{4\tau_2}\,\partial_{t^a}\right), \tag{3.26}$$

is the modular covariant derivative operator mapping modular functions of weight $(\mathfrak{h},\bar{\mathfrak{h}})$ to modular functions of weight $(\mathfrak{h}+2,\bar{\mathfrak{h}})$, whereas the function $\mathcal{F}^{(1)}_{\boldsymbol{p}}$ is given by

$$\mathcal{F}^{(1)}_{\boldsymbol{p}} = \frac{1}{4\pi^2}\sum_{q_\Lambda}\sigma_\gamma\bar\Omega(\gamma)\int_{\ell_\gamma}\mathrm{d}z\,\mathcal{X}^{\rm cl}_\gamma = \frac{\sigma_{\boldsymbol{p}}\, e^{-\pi\tau_2(pt^2)+2\pi\mathrm{i}p^a\tilde c_a}}{4\pi^2\sqrt{2\tau_2(pt^2)}}\,\mathcal{Z}^{(1)}_{\boldsymbol{p}} \tag{3.27}$$

and is proportional to the MSW elliptic genus (2.19). Since in this approximation, $\mathcal{Z}^{(1)}_{\boldsymbol{p}}$ transforms as a modular form of weight $(-\frac{3}{2},\frac{1}{2})$, the contact potential satisfies the required modular properties.

4. Contact potential

Although the complete proof that the D3-instanton corrected hypermultiplet moduli space carries an isometric action of $SL(2,\mathbb{Z})$ requires analyzing the modular transformations of the full system of Darboux coordinates, in this paper we restrict our attention to the modular properties of the contact potential. This provides a highly non-trivial test of S-duality and, as was explained in section 2.4, already has important implications for understanding the modular properties of the elliptic genus and the partition function of DT invariants. The analysis of Darboux coordinates will be presented in [18].

Let us evaluate the D3-instanton contribution to the contact potential (3.10) up to second order in the expansion in DT invariants. This requires the knowledge of the mirror map for u^a to the same order. We assume that it is given by the same relation (3.20) as above where, however, the Fourier modes $\mathcal{X}_\gamma$ should now be substituted by their expressions (3.21) including one-instanton contributions. In appendix C.1 we provide the details of the calculation. To present the final result, we first introduce the obvious generalization of the function (3.27),

$$\mathcal{F} = \frac{1}{4\pi^2} \sum_{\gamma\in\Gamma_+} \sigma_\gamma \bar{\Omega}(\gamma) \int_{\ell_\gamma} \mathrm{d}z\, \mathcal{X}_\gamma. \tag{4.1}$$

Expanding it to the second order, one finds

$$\mathcal{F} = \sum_{\boldsymbol{p}} \mathcal{F}^{(1)}_{\boldsymbol{p}} + \sum_{\boldsymbol{p}_1,\boldsymbol{p}_2} \mathcal{F}^{(2)}_{\boldsymbol{p}_1\boldsymbol{p}_2} + \cdots\,, \tag{4.2}$$

where $\mathcal{F}^{(1)}_{\boldsymbol{p}}$ is given in (3.27), whereas the second order term reads

$$\mathcal{F}^{(2)}_{\boldsymbol{p}_1\boldsymbol{p}_2} = \frac{1}{8\pi^3} \sum_{q_{1,\Lambda},q_{2,\Lambda}} \sigma_{\gamma_1}\sigma_{\gamma_2}\bar{\Omega}(\gamma_1)\bar{\Omega}(\gamma_2) \int_{\ell_{\gamma_1}} \mathrm{d}z_1 \int_{\ell_{\gamma_2}} \mathrm{d}z_2 \left[(tp_1p_2) - \frac{\mathrm{i}\langle\gamma_1,\gamma_2\rangle}{z_2-z_1}\right] \mathcal{X}^{\mathrm{cl}}_{\gamma_1}(z_1)\mathcal{X}^{\mathrm{cl}}_{\gamma_2}(z_2), \tag{4.3}$$

where we used (3.23). The function encoding the contact potential is obtained by *halving* the coefficient of the second order term in the simple twistor integral (4.1),

$$\tilde{\mathcal{F}}_{\boldsymbol{p}} = \mathcal{F}^{(1)}_{\boldsymbol{p}} + \frac{1}{2} \sum_{\boldsymbol{p}_1+\boldsymbol{p}_2=\boldsymbol{p}} \mathcal{F}^{(2)}_{\boldsymbol{p}_1\boldsymbol{p}_2}, \tag{4.4}$$

While this prescription may seem surprising at first sight, it follows from the indistinguishability of the constituents with charges $\boldsymbol{p}_1, \boldsymbol{p}_2$.

In terms of the function (4.4), the D3-instanton contribution to the contact potential has a simple representation which generalizes (3.25),

$$\delta e^{\Phi} = \frac{\tau_2}{2}\,\mathrm{Re} \sum_{\boldsymbol{p}} \mathcal{D}_{-\frac{3}{2}} \tilde{\mathcal{F}}_{\boldsymbol{p}} - \frac{1}{8} \sum_{\boldsymbol{p}_1,\boldsymbol{p}_2} (tp_1p_2)\, \mathcal{F}^{(1)}_{\boldsymbol{p}_1} \overline{\mathcal{F}^{(1)}_{\boldsymbol{p}_2}}. \tag{4.5}$$

Since $\mathcal{F}^{(1)}_{\boldsymbol{p}_1}$ transforms under S-duality with modular weight $(-\frac{3}{2},\frac{1}{2})$[10], it is immediate to see that the last term in (4.5) transforms as a modular form of weight $(-\frac{1}{2},-\frac{1}{2})$, as required for the contact potential. Hence the same should be true for the first term. Since $\mathcal{D}_{-\frac{3}{2}}$ is a modular covariant operator, raising the weight by (2,0), one concludes that modular invariance requires that the full function $\tilde{\mathcal{F}}_{\boldsymbol{p}}$ must be a modular form of weight $(-\frac{3}{2},\frac{1}{2})$.

Let us rewrite this function in terms of MSW invariants and perform its theta series decomposition. To this end, we plug in the expansion of DT invariants given by (2.14), keeping only terms of second order in $\bar{\Omega}^{\mathrm{MSW}}$, and then substitute (3.22). The result is

$$\begin{aligned}
\tilde{\mathcal{F}}_{\boldsymbol{p}} = \frac{e^{-2\pi S^{\mathrm{cl}}_{\boldsymbol{p}}}}{4\pi^2} \Biggl\{ & \sum_{\boldsymbol{\mu}\in\Lambda^\star/\Lambda} h_{\boldsymbol{p},\boldsymbol{\mu}} \sum_{\boldsymbol{k}\in\Lambda_i+\boldsymbol{\mu}+\frac{1}{2}\boldsymbol{p}} \sigma_\gamma \mathcal{Y}_\gamma\, \mathcal{X}^{(\theta)}_{\boldsymbol{p},\boldsymbol{k}} \\
& + \frac{1}{2} \sum_{\boldsymbol{p}_1+\boldsymbol{p}_2=\boldsymbol{p}} \sum_{\boldsymbol{\mu}_i\in\Lambda^\star/\Lambda_i} h_{\boldsymbol{p}_1,\boldsymbol{\mu}_1}\, h_{\boldsymbol{p}_2,\boldsymbol{\mu}_2} \sum_{\boldsymbol{k}_i\in\Lambda_i+\boldsymbol{\mu}_i+\frac{1}{2}\boldsymbol{p}_i} \sigma_{\gamma_1}\sigma_{\gamma_2} \biggl[\langle\gamma_1,\gamma_2\rangle \Delta^{\boldsymbol{t}}_{\gamma_1\gamma_2} \mathcal{Y}_\gamma \mathbf{E}\Bigl(\frac{\mathcal{T}}{2}\,\mathrm{Q}_{\boldsymbol{p}_1,\boldsymbol{p}_2}(\boldsymbol{k}_1,\boldsymbol{k}_2)\Bigr)\, \mathcal{X}^{(\theta)}_{\boldsymbol{p},\boldsymbol{k}} \\
& \quad + \frac{1}{2\pi} \Bigl((tp_1p_2)\, \mathcal{Y}_{\gamma_1}\mathcal{Y}_{\gamma_2} - \mathrm{i}\langle\gamma_1,\gamma_2\rangle\, \mathcal{Y}_{\gamma_1\gamma_2} \Bigr) \biggr] \mathcal{X}^{(\theta)}_{\boldsymbol{p}_1,\boldsymbol{k}_1} \mathcal{X}^{(\theta)}_{\boldsymbol{p}_2,\boldsymbol{k}_2} \Biggr\},
\end{aligned} \tag{4.6}$$

[10]The modular anomaly of $\mathcal{F}^{(1)}_{\boldsymbol{p}_1}$ discussed in section 2 is of second order in the instanton expansion and therefore can be ignored in our approximation in the discussion of the last term in (4.5).

where we defined

$$\mathcal{Y}_\gamma = \int_{\ell_\gamma} \mathrm{d}z\, e^{-2\pi Q_{\gamma_1}(z)}, \qquad \mathcal{Y}_{\gamma_1\gamma_2} = \int_{\ell_{\gamma_1}} \mathrm{d}z_1 \int_{\ell_{\gamma_2}} \mathrm{d}z_2\, \frac{e^{-2\pi\left(Q_{\gamma_1}(z_1)+Q_{\gamma_2}(z_2)\right)}}{z_2-z_1}. \tag{4.7}$$

The first integral is Gaussian and is easily evaluated, whereas the second integral is more involved and computed in appendix C.2. The two results are

$$\mathcal{Y}_\gamma = \frac{1}{\sqrt{2\tau_2(pt^2)}}\,, \qquad \mathcal{Y}_{\gamma_1\gamma_2} = -\frac{\pi\mathrm{i}\, e^{2\pi\tau_2\mathcal{I}^2_{\gamma_1\gamma_2}}}{\sqrt{2\tau_2(pt^2)}}\,\mathrm{sgn}(\mathcal{I}_{\gamma_1\gamma_2})\,\beta_{\frac12}(2\tau_2\mathcal{I}^2_{\gamma_1\gamma_2}). \tag{4.8}$$

Plugging them into (4.6) and taking into account that due to (3.18)

$$\sigma_{\boldsymbol{p}_1}\sigma_{\boldsymbol{p}_2}\sigma^{-1}_{\boldsymbol{p}_1+\boldsymbol{p}_2} = \mathbf{E}\big(A_{ab}p_1^a p_2^b\big) = (-1)^{(p_1^2 p_2)}, \tag{4.9}$$

one finds that $\tilde{\mathcal{F}}_{\boldsymbol{p}}$ coincides, up to a modular invariant factor, with the function given in (2.33). As was shown in section 2, this function is indeed a modular form of weight $(-\frac32,\frac12)$ provided $h_{\boldsymbol{p},\boldsymbol{\mu}}$ is mock modular and its modular completion is given by (2.30) and (1.3). In this case, in the second order approximation, it can be identified with the partition function (2.31) so that we arrive at

$$\tilde{\mathcal{F}}_{\boldsymbol{p}} = \frac{\sigma_{\boldsymbol{p}}\, e^{-2\pi S^{\rm cl}_{\boldsymbol{p}}}}{4\pi^2\sqrt{2\tau_2(pt^2)}}\,\widehat{\mathcal{Z}}_{\boldsymbol{p}}\,. \tag{4.10}$$

This result generalizes (3.27) and ensures the right transformation properties of the contact potential.

5. Discussion

In this paper we studied the invariance of the D3-instanton corrected metric on the hypermultiplet moduli space in type IIB Calabi-Yau string vacua under S-duality. We restricted ourselves to the two-instanton, large volume approximation and concentrated on the contact potential e^Φ, which is sensitive to all quantum corrections to the metric. S-duality requires that it must transform as a modular form of fixed weight. We showed that in our approximation, D3-instanton contributions to e^Φ can be expressed in terms of the modular derivative of a function $\widehat{\mathcal{Z}}_{\boldsymbol{p}}$ constructed, on the one hand, from a non-holomorphic modification $\widehat{h}_{\boldsymbol{p},\boldsymbol{\mu}}$ (2.30) of the generating function of the MSW invariants $h_{\boldsymbol{p},\boldsymbol{\mu}}$, and on the other hand, from the double theta series $\widehat{\Psi}$ constructed in [16]. The modular invariance of e^Φ then requires that $\widehat{\mathcal{Z}}_{\boldsymbol{p}}$ should transform as a Jacobi form (of fixed weight and multiplier system), which in turn implies that $\widehat{h}_{\boldsymbol{p},\boldsymbol{\mu}}$ should transform as a vector-valued modular form. Thus, when the divisor $\mathcal{D}$ wrapped by the D3-brane is reducible, $h_{\boldsymbol{p},\boldsymbol{\mu}}$ is only mock modular. In the case when $\mathcal{D}$ can split into at most two effective divisors $\mathcal{D}_1+\mathcal{D}_2$, the modular anomaly is dictated by the non-holomorphic completion $R_{\boldsymbol{p},\boldsymbol{\mu}}$ given in (1.3). Clearly, it would be desirable to find independent checks of this conjecture. Beyond this, our work opens several avenues for future research:

- The modular function $\widehat{\mathcal{Z}}_{\boldsymbol{p}}$ (1.2) can be viewed as the BPS partition function of D4-D2-D0 black holes in $\mathcal{N}=2$ supergravity in $\mathbb{R}^{3,1}$. The non-holomorphic terms which are necessary for modularity, can be understood as a consequence of the continuum of scattering states in $\mathbb{R}^{3,1}$. It would be interesting to derive these non-holomorphic terms from the space-time perspective along the lines of [30, 60].

- As was done in (2.41), the first term $\widehat{\mathcal{Z}}_{\boldsymbol{p}}^{(1)}$ in (1.2) should be interpreted as the completed elliptic genus $\widehat{\chi}_{\boldsymbol{p}}$ of the MSW SCFT obtained by wrapping an M5-brane on $\mathcal{D}$. The holomorphic anomaly is expected to arise from a spectral asymmetry in a continuum of states, corresponding to configurations where the M5-brane on $\mathcal{D}$ splits into two M5-branes wrapping $\mathcal{D}_1$ and $\mathcal{D}_2$. It would be interesting to derive the modular anomaly of the generating function $h_{\boldsymbol{p},\boldsymbol{\mu}}$ using this interpretation, which would require a better understanding of the M5-brane CFT [11, 61].

- Another challenging approach is to determine the DT invariants $\Omega(\gamma;\boldsymbol{z})$ for specific Calabi-Yau manifolds using enumerative algebraic geometry. Since our results give an explicit constraint on DT invariants for pure codimension 2 sheaves, the knowledge of $\Omega(\gamma;\boldsymbol{z})$ will allow to verify the necessity of the non-holomorphic modification $R_{\boldsymbol{p},\boldsymbol{\mu}}$. One could, for example, try to extend the approach of [13] to two D4-branes on the quintic, or to Calabi-Yau threefolds with $b_2 > 1$.

- Returning to the subject of hypermultiplet moduli spaces, the fact that the contact potential is modular is only a necessary condition for the existence of an isometric action of S-duality on $\mathcal{M}_H$. A complete proof requires analyzing the Darboux coordinates and showing that they transform by a holomorphic contact transformation, as was done in the one-instanton approximation in [12]. A similar analysis at two-instanton level will be the subject of the subsequent work [18].

- It would also be interesting to go beyond the large volume approximation and to arbitrary order in the multi-instanton expansion. An important step in this direction would be to obtain a twistorial formulation of D3-instantons which is manifestly invariant under S-duality, along the lines outlined in [58].

- Finally, a far-reaching goal is to get a complete non-perturbative description of the exact hypermultiplet moduli space including, in particular, five-brane instantons. Their modularity can be enforced by covariantizing the known results on D5-instantons under S-duality. This was realized at a linearized level in [38] and attempted beyond the linear approximation in [55, 47], but it is not clear whether a simple covariantization is sufficient to get the complete picture.

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A. Indefinite theta series

In this section, we provide an alternative proof to the fact that $\widehat{\Psi} = \Psi + \Psi^{(+)} + \Psi^{(-)}$ transforms as a vector valued Jacobi form of weight $(b_2 + \frac{1}{2}, \frac{1}{2})$. The original alternative proof, which can be found in Proposition 4 of [16], invokes the standard Poisson resummation technique to establish the modular transformation property of the theta series. Instead, our proof is based on the use of Vignéras' theorem [44]. This theorem can be used to prove modularity for a quite general class of theta series, and will play a crucial role in the study of Darboux coordinates relegated to [18].

A.1 Vignéras' theorem

Let us start by stating Vignéras' theorem in a general fashion. Let Λ be an n-dimensional lattice equipped with a bilinear form $B(\boldsymbol{x}, \boldsymbol{y}) \equiv \boldsymbol{x} \cdot \boldsymbol{y}$, where $\boldsymbol{x}, \boldsymbol{y} \in \Lambda \otimes \mathbb{R}$, such that its associated quadratic form has signature (n_+, n_-) and is integer valued, i.e. $\boldsymbol{k}^2 \in \mathbb{Z}$ for $\boldsymbol{k} \in \Lambda$. Furthermore, let $\boldsymbol{p} \in \Lambda$ be a characteristic vector (such that $\boldsymbol{k}^2 + \boldsymbol{k} \cdot \boldsymbol{p} \in 2\mathbb{Z}$, $\forall\, \boldsymbol{k} \in \Lambda$), $\boldsymbol{\mu} \in \Lambda^*/\Lambda$ a glue vector, and λ an arbitrary integer. With the usual notation $q = \mathbf{E}(\tau)$, we consider the following family of theta series

$$\vartheta_{\boldsymbol{p},\boldsymbol{\mu}}(\Phi, \lambda; \tau, \boldsymbol{b}, \boldsymbol{c}) = \tau_2^{-\lambda/2} \sum_{\boldsymbol{k} \in \Lambda + \boldsymbol{\mu} + \frac{1}{2}\boldsymbol{p}} (-1)^{\boldsymbol{k}\cdot\boldsymbol{p}}\, \Phi(\sqrt{2\tau_2}(\boldsymbol{k} + \boldsymbol{b}))\, q^{-\frac{1}{2}(\boldsymbol{k}+\boldsymbol{b})^2}\, \mathbf{E}\big(\boldsymbol{c} \cdot (\boldsymbol{k} + \tfrac{1}{2}\boldsymbol{b})\big) \tag{A.1}$$

defined by the kernel $\Phi(\boldsymbol{x})$. Irrespective of the choice of this kernel and the parameter λ, any such theta series satisfies the following elliptic properties

$$\begin{aligned} \vartheta_{\boldsymbol{p},\boldsymbol{\mu}}(\Phi, \lambda; \tau, \boldsymbol{b} + \boldsymbol{k}, \boldsymbol{c}) =& (-1)^{\boldsymbol{k}\cdot\boldsymbol{p}}\, \mathbf{E}\big(-\tfrac{1}{2}\, \boldsymbol{c} \cdot \boldsymbol{k}\big)\, \vartheta_{\boldsymbol{p},\boldsymbol{\mu}}(\Phi, \lambda; \tau, \boldsymbol{b}, \boldsymbol{c}), \\ \vartheta_{\boldsymbol{p},\boldsymbol{\mu}}(\Phi, \lambda; \tau, \boldsymbol{b}, \boldsymbol{c} + \boldsymbol{k}) =& (-1)^{\boldsymbol{k}\cdot\boldsymbol{p}}\, \mathbf{E}\big(\tfrac{1}{2}\, \boldsymbol{b} \cdot \boldsymbol{k}\big)\, \vartheta_{\boldsymbol{p},\boldsymbol{\mu}}(\Phi, \lambda; \tau, \boldsymbol{b}, \boldsymbol{c}). \end{aligned} \tag{A.2}$$

Now let us require that in addition the kernel satisfies the following two conditions:

1. Let $D(\boldsymbol{x})$ be any differential operator of order ≤ 2, and $R(\boldsymbol{x})$ any polynomial of degree ≤ 2. Then $f(\boldsymbol{x}) = \Phi(\boldsymbol{x})\, e^{\pi \boldsymbol{x}^2/2}$ must be such that $f(\boldsymbol{x})$, $D(\boldsymbol{x}) f(\boldsymbol{x})$ and $R(\boldsymbol{x}) f(\boldsymbol{x}) \in L^2(\Lambda \otimes \mathbb{R}) \bigcap L^1(\Lambda \otimes \mathbb{R})$.

2. $\Phi(\boldsymbol{x})$ must satisfy
$$\left[\partial_{\boldsymbol{x}}^2 + 2\pi \boldsymbol{x} \cdot \partial_{\boldsymbol{x}}\right] \Phi(\boldsymbol{x}) = 2\pi\lambda\, \Phi(\boldsymbol{x}). \tag{A.3}$$

Then the theta series (A.1) transforms as a Jacobi form of weight $(\lambda + n/2, 0)$. Explicitly the modular transformation properties are given by

$$\begin{aligned} \vartheta_{\boldsymbol{p},\boldsymbol{\mu}}(\Phi, \lambda; -1/\tau, \boldsymbol{c}, -\boldsymbol{b}) &= \frac{(-\mathrm{i}\tau)^{\lambda + \frac{n}{2}}}{\sqrt{|\Lambda^*/\Lambda|}}\, \mathbf{E}\big(\tfrac{1}{4}\, \boldsymbol{p}^2\big) \sum_{\boldsymbol{\nu} \in \Lambda^*/\Lambda} \mathbf{E}(\boldsymbol{\mu} \cdot \boldsymbol{\nu})\, \vartheta_{\boldsymbol{p},\boldsymbol{\mu}}(\Phi, \lambda; \tau, \boldsymbol{b}, \boldsymbol{c}), \\ \vartheta_{\boldsymbol{p},\boldsymbol{\mu}}(\Phi, \lambda; \tau + 1, \boldsymbol{b}, \boldsymbol{c} + \boldsymbol{b})) &= \mathbf{E}\big(-\tfrac{1}{2}(\boldsymbol{\mu} + \tfrac{1}{2}\boldsymbol{p})^2\big)\, \vartheta_{\boldsymbol{p},\boldsymbol{\mu}}(\Phi, \lambda; \tau, \boldsymbol{b}, \boldsymbol{c}). \end{aligned} \tag{A.4}$$

For a positive definite lattice (with $n_- = 0$), the theta series $\vartheta_{\boldsymbol{p},\boldsymbol{\mu}}$ is related to the standard one with complex elliptic variables $\boldsymbol{v} \in \mathbb{C}^{n_+}$ under the change of variables

$$\vartheta_{\boldsymbol{p},\boldsymbol{\mu}}(\tau, \boldsymbol{b}, \boldsymbol{c}) = e^{\mathrm{i}\pi(\tau \boldsymbol{b}^2 - \boldsymbol{b}\cdot\boldsymbol{c})}\, \tilde{\vartheta}_{\boldsymbol{p},\boldsymbol{\mu}}(\tau, \boldsymbol{v} = \boldsymbol{b}\tau - \boldsymbol{c}). \tag{A.5}$$

One can check that the covariant derivatives preserve the form of the theta series (A.1) changing the kernel and parameter λ according to

$$\begin{aligned} \tau_2^2\partial_{\bar\tau} \ &: (\Phi,\lambda)\mapsto \left(\frac{\mathrm{i}}{4}\left(\boldsymbol{x}\partial_{\boldsymbol{x}}\Phi-\lambda\Phi\right),\lambda-2\right), \\ \partial_\tau-\frac{\mathrm{i}(\lambda+\frac{n}{2})}{2\tau_2} \ &: (\Phi,\lambda)\mapsto \left(-\frac{\mathrm{i}}{4}\left(\boldsymbol{x}\partial_{\boldsymbol{x}}\Phi+(\lambda+n+2\pi\boldsymbol{x}^2)\Phi\right),\lambda+2\right). \end{aligned} \tag{A.6}$$

A.2 Examples: Siegel-Narain and Zwegers' theta series

Let us now restrict to the case of signature $(1,n-1)$. A useful class of solutions of (A.3) are functions of one variable, the projection of $\boldsymbol{x}$ on a fixed time-like vector $\boldsymbol{t}$ with $\boldsymbol{t}^2>0$,

$$\Phi(\boldsymbol{x})=f(x_+^{(\boldsymbol{t})}),\qquad x_+^{(\boldsymbol{t})}=\frac{\boldsymbol{x}\cdot\boldsymbol{t}}{\sqrt{\boldsymbol{t}^2}},\qquad f''+2\pi(x_+^{(\boldsymbol{t})}f'-\lambda f)=0. \tag{A.7}$$

The solution $f=e^{-\pi(x_+^{(\boldsymbol{t})})^2}$ with $\lambda=-1$ gives, up to a factor of $\tau_2^{1/2}$, the standard Siegel-Narain theta series (2.22), a Jacobi form of weight $(\frac{n-1}{2},\frac12)$.

Another solution of (A.3) with $\lambda=0$ is provided by the function $f=E(x_+^{(\boldsymbol{t})})$ with $E(x)=\mathrm{Erf}(\sqrt{\pi}x)$. However, since $E(x)\to\mathrm{sgn}(x)$ as $x\to\infty$, it does not satisfy the decay conditions. For two time-like vectors $\boldsymbol{t},\boldsymbol{t}'$, however, the difference $f=E(x_+^{(\boldsymbol{t})})-E(x_+^{(\boldsymbol{t}')})$ does. This leads to Zwegers indefinite theta series [19] of weight $(\frac{n}{2},0)$,

$$\begin{aligned} \widehat\Theta_{\boldsymbol{p},\boldsymbol{\mu}}(\tau,\boldsymbol{t},\boldsymbol{t}',\boldsymbol{b},\boldsymbol{c})=&\sum_{\boldsymbol{k}\in\Lambda+\boldsymbol{\mu}+\frac12\boldsymbol{p}}(-1)^{\boldsymbol{k}\cdot\boldsymbol{p}}\left[\mathrm{Erf}\left(\sqrt{2\pi\tau_2}(k+b)_+^{(\boldsymbol{t})}\right)-\mathrm{Erf}\left(\sqrt{2\pi\tau_2}(k+b)_+^{(\boldsymbol{t}')}\right)\right] \\ &\times q^{-\frac12(\boldsymbol{k}+\boldsymbol{b})^2}\,\mathbf{E}\big(\boldsymbol{c}\cdot(\boldsymbol{k}+\tfrac12\boldsymbol{b})\big)\,, \end{aligned} \tag{A.8}$$

which provides the modular completion of the holomorphic indefinite theta series

$$\begin{aligned} \Theta_{\boldsymbol{p},\boldsymbol{\mu}}(\tau,\boldsymbol{t},\boldsymbol{t}',\boldsymbol{b},\boldsymbol{c})=&\sum_{\boldsymbol{k}\in\Lambda+\boldsymbol{\mu}+\frac12\boldsymbol{p}}(-1)^{\boldsymbol{k}\cdot\boldsymbol{p}}\left[\mathrm{sgn}((k+b)_+^{(\boldsymbol{t})})-\mathrm{sgn}((k+b)_+^{(\boldsymbol{t}')})\right] \\ &\times q^{-\frac12(\boldsymbol{k}+\boldsymbol{b})^2}\,\mathbf{E}\big(\boldsymbol{c}\cdot(\boldsymbol{k}+\tfrac12\boldsymbol{b})\big)\,. \end{aligned} \tag{A.9}$$

A.3 Construction of $\widehat\Psi$

Let us now consider a variant of (A.9), with an extra insertion of $(k+b)_+^{(\boldsymbol{t})}$ in the sum:

$$\begin{aligned} \Theta'_{\boldsymbol{p},\boldsymbol{\mu}}(\tau,\boldsymbol{t},\boldsymbol{t}',\boldsymbol{b},\boldsymbol{c})=&\sum_{\boldsymbol{k}\in\Lambda+\boldsymbol{\mu}+\frac12\boldsymbol{p}}(-1)^{\boldsymbol{k}\cdot\boldsymbol{p}}\left[\mathrm{sgn}((k+b)_+^{(\boldsymbol{t})})-\mathrm{sgn}((k+b)_+^{(\boldsymbol{t}')})\right](k+b)_+^{(\boldsymbol{t})} \\ &\times\, q^{-\frac12(\boldsymbol{k}+\boldsymbol{b})^2}\,\mathbf{E}\big(\boldsymbol{c}\cdot(\boldsymbol{k}+\tfrac12\boldsymbol{b})\big)\,. \end{aligned} \tag{A.10}$$

To find the modular completion of (A.10), we need to find a solution of (A.3) which asymptotes to

$$x_+^{(\boldsymbol{t})}\left[\mathrm{sgn}(x_+^{(\boldsymbol{t})})-\mathrm{sgn}(x_+^{(\boldsymbol{t}')})\right]. \tag{A.11}$$

The first term $|x_+^{(\boldsymbol{t})}|$ can be promoted to $F(x_+^{(\boldsymbol{t})})$ where

$$F(x)=x\,\mathrm{Erf}(\sqrt{\pi}x)+\frac{1}{\pi}\,e^{-\pi x^2}, \tag{A.12}$$

which is a solution of (A.7) with $\lambda = 1$. To deal with the second term, we decompose $\boldsymbol{t}$ into its projection on $\boldsymbol{t}'$ and its orthogonal complement:

$$\boldsymbol{t} = \frac{\boldsymbol{t}\cdot\boldsymbol{t}'}{\boldsymbol{t}'\cdot\boldsymbol{t}'}\,\boldsymbol{t}' + \left[\boldsymbol{t} - \frac{\boldsymbol{t}\cdot\boldsymbol{t}'}{\boldsymbol{t}'\cdot\boldsymbol{t}'}\,\boldsymbol{t}'\right]. \tag{A.13}$$

Contracting with $\boldsymbol{x}/\sqrt{\boldsymbol{t}^2}$ and multiplying by $\mathrm{sgn}(x_+^{(\boldsymbol{t}')})$, one obtains

$$x_+^{(\boldsymbol{t})}\mathrm{sgn}(x_+^{(\boldsymbol{t}')}) = \frac{\boldsymbol{t}\cdot\boldsymbol{t}'}{\sqrt{\boldsymbol{t}^2\boldsymbol{t}'^2}}\,|x_+^{(\boldsymbol{t}')}| + \left[x_+^{(\boldsymbol{t})} - \frac{\boldsymbol{t}\cdot\boldsymbol{t}'}{\sqrt{\boldsymbol{t}^2\boldsymbol{t}'^2}}\,x_+^{(\boldsymbol{t}')}\right]\mathrm{sgn}(x_+^{(\boldsymbol{t}')}). \tag{A.14}$$

The first term can be promoted to $F(x_+^{(\boldsymbol{t}')})$, while in the second term, $\mathrm{sgn}(x_+^{(\boldsymbol{t}')})$ can be promoted to $E(x_+^{(\boldsymbol{t}')})$. Thus, a solution of the Vignéras' equation with the required decay properties can be obtained by promoting (A.11) to

$$\begin{aligned}\Phi(\boldsymbol{x}) =& F(x_+^{(\boldsymbol{t})}) - \frac{\boldsymbol{t}\cdot\boldsymbol{t}'}{\sqrt{\boldsymbol{t}^2\boldsymbol{t}'^2}}\,F(x_+^{(\boldsymbol{t}')}) - \left[x_+^{(\boldsymbol{t})} - \frac{\boldsymbol{t}\cdot\boldsymbol{t}'}{\sqrt{\boldsymbol{t}^2\boldsymbol{t}'^2}}\,x_+^{(\boldsymbol{t}')}\right]E(x_+^{(\boldsymbol{t}')})\\ =& x_+^{(\boldsymbol{t})}\left[E(x_+^{(\boldsymbol{t})}) - E(x_+^{(\boldsymbol{t}')})\right] + \frac{1}{\pi}\,e^{-\pi(x_+^{(\boldsymbol{t})})^2} - \frac{\boldsymbol{t}\cdot\boldsymbol{t}'}{\pi\sqrt{\boldsymbol{t}^2\boldsymbol{t}'^2}}\,e^{-\pi(x_+^{(\boldsymbol{t}')})^2}.\end{aligned} \tag{A.15}$$

By construction, this is a solution of (A.3) with $\lambda = 1$ and thus, using this kernel in (A.1), one obtains a modular completion of (A.10) with weight $(\frac{n}{2}+1, 0)$. Note that unlike the case of Zwegers' theta series, the difference between (A.15) and (A.11) is *not* the difference of a function of $\boldsymbol{t}$ and a function of $\boldsymbol{t}'$.

We now apply this construction to produce the double theta series constructed in [16]. We start with a lattice $\Lambda\oplus\Lambda$ which carries the quadratic form $\mathcal{K}^2 = \kappa_{abc}k_1^a k_1^b p_1^c + \kappa_{abc}k_2^a k_2^b p_2^c$ of signature $(2, 2b_2-2)$, where $\mathcal{K} = (\boldsymbol{k}_1, \boldsymbol{k}_2)$ and $\boldsymbol{p}_1, \boldsymbol{p}_2$ are two vectors in Λ with $p_1^3, p_2^3 > 0$. Let $\boldsymbol{p} = \boldsymbol{p}_1 + \boldsymbol{p}_2$ and $\boldsymbol{t}\in\Lambda\otimes\mathbb{R}$, such that t^3, (p_1t^2), (p_2t^2), (p_1p_2t) are all positive. Then it is easy to check that the vectors

$$\mathcal{T} = \frac{(t^a, t^a)}{\sqrt{(pt^2)}}, \qquad \mathcal{P} = \frac{(p_2^a, -p_1^a)}{\sqrt{(pp_1p_2)}}, \qquad \mathcal{P}' = \frac{((p_2t^2)t^a, -(p_1t^2)t^a)}{\sqrt{(p_1t^2)(p_2t^2)(pt^2)}} \tag{A.16}$$

satisfy

$$\begin{aligned}&\mathcal{T}^2 = \mathcal{P}^2 = \mathcal{P}'^2 = 1, \qquad \mathcal{T}\cdot\mathcal{P} = \mathcal{T}\cdot\mathcal{P}' = 0, \qquad \mathcal{P}\cdot\mathcal{P}' = \sqrt{\frac{(pt^2)(p_1p_2t)^2}{(p_1t^2)(p_2t^2)(pp_1p_2)}},\\ &\qquad \mathcal{K}\cdot\mathcal{P} = \frac{\langle\gamma_1,\gamma_2\rangle}{\sqrt{(pp_1p_2)}}, \qquad \mathcal{K}\cdot\mathcal{P}' = \mathcal{I}_{\gamma_1\gamma_2}(\boldsymbol{t}, \boldsymbol{b}=0),\end{aligned} \tag{A.17}$$

where in the last two relations, as usual, we took the charge vectors as $\gamma_i = (0, p_i^a, \kappa_{abc}k_i^b p_i^c, q_{i,0})$.

Now we want to use the kernel (A.15) with the above quadratic form and vectors $\boldsymbol{t}, \boldsymbol{t}'$ identified with $\mathcal{P}, \mathcal{P}'$, respectively. However, the two cases differ by the signature of the quadratic form: due to the additional positive direction, the naive use of (A.15) with the above data leads to a kernel which does not decay in the direction described by the vector $\mathcal{T}$. Fortunately, the situation can be cured by multiplying by an additional exponential factor where the charge vector is projected on $\mathcal{T}$, as in the Siegel-Narain theta series. In this way we arrive to the following kernel

$$\Phi(\mathcal{K}) = e^{-\pi(\mathcal{K}\cdot\mathcal{T})^2}\left[(\mathcal{K}\cdot\mathcal{P})\Big(E(\mathcal{K}\cdot\mathcal{P}) - E(\mathcal{K}\cdot\mathcal{P}')\Big) + \frac{e^{-\pi(\mathcal{K}\cdot\mathcal{P})^2}}{\pi} - \frac{(\mathcal{P}\cdot\mathcal{P}')\, e^{-\pi(\mathcal{K}\cdot\mathcal{P}')^2}}{\pi}\right]. \quad (A.18)$$

This is a solution of Vignéras' equation with $\lambda = 0$, which follows from the fact that the two factors separately satisfy this equation with $\lambda = -1$ and $\lambda = 1$, respectively, and that $\mathcal{T}$ is orthogonal to both $\mathcal{P}$ and $\mathcal{P}'$. Furthermore, $\Phi(\mathcal{K})$ satisfies the required decay condition and thus it generates a Jacobi form of weight $(b_2, 0)$,

$$\widehat{\Theta}_{\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2} = \sum_{\boldsymbol{k}_i\in\Lambda_i+\boldsymbol{\mu}_i+\frac12\boldsymbol{p}_i} (-1)^{\boldsymbol{k}_1\cdot\boldsymbol{p}_1+\boldsymbol{k}_2\cdot\boldsymbol{p}_2}\,\Phi(\sqrt{2\tau_2}(\mathcal{K}+\mathcal{B}))\, q^{-\frac12(\mathcal{K}+\mathcal{B})^2}\,\mathbf{E}\big(\mathcal{C}\cdot(\mathcal{K}+\tfrac12\mathcal{B})\big)\,, \quad (A.19)$$

where $\mathcal{B} = (\boldsymbol{b},\boldsymbol{b})$ and $\mathcal{C} = (\boldsymbol{c},\boldsymbol{c})$. Using (A.17), it is straightforward to check that this double theta series reproduces $\widehat{\Psi}_{\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}$ introduced in section 2.2, up to an overall factor of $-(-1)^{(p_1^2p_2)}\sqrt{8\tau_2/(pp_1p_2)}$. This completes the alternative proof that this function is a modular form.

B. Computing $R_{p,\mu}$

In this appendix we derive the explicit expression for the non-holomorphic completion $R_{\boldsymbol{p},\boldsymbol{\mu}}$ of the holomorphic mock modular form $h_{\boldsymbol{p},\boldsymbol{\mu}}$. Our starting point is (2.34), where $\Psi^{(-)}_{\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}$ is defined in (2.26), (2.28). First, we note the identity

$$\mathcal{I}^2_{\gamma_1\gamma_2} = (\boldsymbol{q}_1+\boldsymbol{b})^2_{1+} + (\boldsymbol{q}_2+\boldsymbol{b})^2_{2+} - (\boldsymbol{q}_1+\boldsymbol{q}_2+\boldsymbol{b})^2_{+} \quad (B.1)$$

where the index 1 or 2 denotes the charge used to define the quadratic form, while the index $+$ denotes as usual the projection on the Kähler modulus $\boldsymbol{t}$. For instance, $(\boldsymbol{k})^2_{i+} = \frac{(ktp_i)^2}{(p_it^2)}$. This identity can be used to rewrite the r.h.s. of (2.34) as

$$-\frac{1}{4\pi}\sum_{\substack{\gamma_1,\gamma_2\in\Gamma_+\\ \boldsymbol{p}_1+\boldsymbol{p}_2=\boldsymbol{p}}} (-1)^{\boldsymbol{p}_1\cdot\boldsymbol{q}_1+\boldsymbol{p}_2\cdot\boldsymbol{q}_2+(p_1^2p_2)}|\langle\gamma_1,\gamma_2\rangle|\,\bar\Omega_{\boldsymbol{p}_1,\boldsymbol{\mu}_1}(\hat q_{0,1})\bar\Omega_{\boldsymbol{p}_2,\boldsymbol{\mu}_2}(\hat q_{0,2})\,\beta_{\frac32}\Big(\tfrac{2\tau_2}{(pp_1p_2)}\,\langle\gamma_1,\gamma_2\rangle^2\Big)$$
$$\times\,\mathbf{E}\Big(-\tau(\hat q_{1,0}+\hat q_{2,0}) + \frac{\tau}{2}\Big[(\boldsymbol{q}_1+\boldsymbol{q}_2+\boldsymbol{b})^2 - (\boldsymbol{q}_1+\boldsymbol{b})^2_1 - (\boldsymbol{q}_2+\boldsymbol{b})^2_2\Big]\Big)\,\mathcal{X}^{(\theta)}_{\boldsymbol{p},\boldsymbol{q}_1+\boldsymbol{q}_2}. \quad (B.2)$$

Next, we decompose each of the charges $\boldsymbol{q}_1$, $\boldsymbol{q}_2$ and $\boldsymbol{q} = \boldsymbol{q}_1+\boldsymbol{q}_2$ according to (2.7),

$$\begin{aligned} q_{1,a} =&\mu_{1,a} + \frac12\kappa_{abc}p_1^bp_1^c + \kappa_{abc}p_1^b\epsilon_1^c,\\ q_{2,a} =&\mu_{2,a} + \frac12\kappa_{abc}p_2^bp_2^c + \kappa_{abc}p_2^b\epsilon_2^c,\\ q_a =&\mu_a + \frac12\kappa_{abc}p^bp^c + \kappa_{abc}p^b\epsilon^c, \end{aligned} \quad (B.3)$$

where $\boldsymbol{\epsilon}_i, \boldsymbol{\epsilon}\in\Lambda$ and $\tilde{\boldsymbol{\mu}}_i\in\Lambda^\star/\Lambda_i$, $\tilde{\boldsymbol{\mu}}\in\Lambda^\star/\Lambda$. The sum over charges in (B.2) is then

$$\sum_{\substack{\gamma_1,\gamma_2\in\Gamma_+\\ \boldsymbol{p}_1+\boldsymbol{p}_2=\boldsymbol{p}}} = \sum_{\boldsymbol{p}_1+\boldsymbol{p}_2=\boldsymbol{p}}\,\sum_{\hat q_{i,0}}\,\sum_{\boldsymbol{\mu}_i\in\Lambda^\star/\Lambda_i}\,\sum_{\boldsymbol{\epsilon}_i\in\Lambda_i}. \quad (B.4)$$

Our goal is to exchange the sum over $\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2$ for a sum over $\boldsymbol{\epsilon}, \boldsymbol{\mu}$ and the variable $\boldsymbol{\rho}$ defined by

$$\kappa_{abc} p_1^b \epsilon_1^c = \kappa_{abc} p_1^b \tilde{\epsilon}^c + \rho_a \,, \qquad \kappa_{abc} p_2^b \epsilon_2^c = \kappa_{abc} p_2^b \tilde{\epsilon}^c - \rho_a \,. \tag{B.5}$$

Here $\tilde{\boldsymbol{\epsilon}}$ is a non-integer vector which will be related to $\boldsymbol{\epsilon}$ momentarily. The equations (B.5) uniquely determine the pair $(\tilde{\boldsymbol{\epsilon}}, \boldsymbol{\rho})$ for each pair $(\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2)$. Next, we apply the standard decomposition to the sum of these two equations,

$$\kappa_{abc} p_1^b \epsilon_1^c + \kappa_{abc} p_2^b \epsilon_2^c = \kappa_{abc} p^b \tilde{\epsilon}^c = \tilde{\mu}_a + \kappa_{abc} p^b \epsilon^c \,, \tag{B.6}$$

where $\tilde{\boldsymbol{\mu}} \in \Lambda^\star/\Lambda$. Thus $\tilde{\boldsymbol{\epsilon}}$ is related to $\boldsymbol{\epsilon}$ by

$$\tilde{\epsilon}^a = \epsilon^a + \kappa^{ab} \tilde{\mu}_b \equiv \epsilon^a + \tilde{\mu}^a \,. \tag{B.7}$$

Using (B.6) in (B.3), we find that $\tilde{\boldsymbol{\mu}}$ is related to $\boldsymbol{\mu}$ by

$$\tilde{\mu}_a = \mu_a - \mu_{1,a} - \mu_{2,a} + \kappa_{abc} p_1^b p_2^c. \tag{B.8}$$

As a result, we can now exchange the two vectors $\boldsymbol{\epsilon}_i \in \Lambda_i$ for three variables: $\boldsymbol{\epsilon} \in \Lambda$, $\boldsymbol{\mu} \in \Lambda^\star/\Lambda$ and $\boldsymbol{\rho}$. As can be seen from (B.5) and (B.7), the latter is such that

$$\rho_a + \kappa_{abc} p_1^b \tilde{\mu}^c \in \Lambda_1, \qquad \rho_a - \kappa_{abc} p_2^b \tilde{\mu}^a \in \Lambda_2. \tag{B.9}$$

Thus, one has

$$\sum_{\boldsymbol{\epsilon}_i \in \Lambda_i} = \sum_{\boldsymbol{\mu} \in \Lambda^\star/\Lambda} \ \sum_{\boldsymbol{k} \in \Lambda + \boldsymbol{\mu} + \frac{1}{2}\boldsymbol{p}} \ \sum_{\boldsymbol{\rho} \in (\Lambda_1 - \tilde{\boldsymbol{\mu}}) \cap (\Lambda_2 + \tilde{\boldsymbol{\mu}})} . \tag{B.10}$$

Furthermore, let us substitute the decomposition of $\boldsymbol{q}_i$ in terms of the new variables into the combinations of charges appearing in (B.2). There are three such combinations:

- the symplectic product of two charges

$$\mathrm{S}_{\boldsymbol{p}_1,\boldsymbol{p}_2}(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\rho}) \equiv \langle \gamma_1, \gamma_2 \rangle = p_2^a \mu_{1,a} - p_1^a \mu_{2,a} + \frac{1}{2}\, \kappa_{abc} p_1^a p_2^b (p_2^c - p_1^c) + p^a \rho_a; \tag{B.11}$$

- the square bracket in the exponential

$$(\boldsymbol{q}_1 + \boldsymbol{q}_2 + \boldsymbol{b})^2 - (\boldsymbol{q}_1 + \boldsymbol{b})_1^2 - (\boldsymbol{q}_2 + \boldsymbol{b})_2^2 = \mathrm{Q}_{\boldsymbol{p}_1,\boldsymbol{p}_2}(\boldsymbol{\nu}_1, \boldsymbol{\nu}_2) \,, \tag{B.12}$$

  where the quadratic form $\mathrm{Q}_{\boldsymbol{p}_1,\boldsymbol{p}_2} : \Lambda^* \oplus \Lambda^* \to \mathbb{Q}$ is defined by

$$\mathrm{Q}_{\boldsymbol{p}_1,\boldsymbol{p}_2}(\boldsymbol{q}_1, \boldsymbol{q}_2) = (\boldsymbol{q}_1 + \boldsymbol{q}_2)^2 - (\boldsymbol{q}_1)_1^2 - (\boldsymbol{q}_2)_2^2 \tag{B.13}$$

  and

$$\begin{aligned} \nu_{1,a} &= \mu_{1,a} + \frac{1}{2}\, \kappa_{abc} p_1^b (p_1^c + \tilde{\mu}^c) + \rho_a, \\ \nu_{2,a} &= \mu_{2,a} + \frac{1}{2}\, \kappa_{abc} p_2^b (p_2^c + \tilde{\mu}^c) - \rho_a; \end{aligned} \tag{B.14}$$

- the sign factor

$$(-1)^{\boldsymbol{p}_1 \cdot \boldsymbol{q}_1 + \boldsymbol{p}_2 \cdot \boldsymbol{q}_2 + (p_1^2 p_2)} = (-1)^{\boldsymbol{k} \cdot \boldsymbol{p}} (-1)^{\mathrm{S}_{\boldsymbol{p}_1,\boldsymbol{p}_2}(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\rho})}. \tag{B.15}$$

Note that the only dependence on $\boldsymbol{k}$ appears in the sign factor and in $\mathcal{X}^{(\theta)}_{\boldsymbol{p},\boldsymbol{k}}$. Thus, the corresponding sum produces the theta series (2.22) and one obtains

$$\sum_{\boldsymbol{\mu}\in\Lambda^\star/\Lambda} R_{\boldsymbol{p},\boldsymbol{\mu}}(\tau)\,\theta_{\boldsymbol{p},\boldsymbol{\mu}}(\tau,\boldsymbol{t},\boldsymbol{b},\boldsymbol{c}) \tag{B.16}$$

with $R_{\boldsymbol{p},\boldsymbol{\mu}}(\tau)$ given in (1.3).

It is important to check that the sum (1.3) is convergent. To this aim, note that for large x, $\beta_{\frac{3}{2}}(x) < e^{-\pi x}$. Thus, we need to show that

$$(\boldsymbol{\rho})_1^2 + (\boldsymbol{\rho})_2^2 - \frac{2(\boldsymbol{p}\cdot\boldsymbol{\rho})^2}{(pp_1p_2)} < 0. \tag{B.17}$$

Defining $\boldsymbol{k}_1, \boldsymbol{k}_2$ via $\rho_a = \kappa_{abc}p_1^b k_1^c = -\kappa_{abc}p_2^b k_2^c$, (B.17) is equivalent to

$$\left[(k_1^2p_1) - \frac{(k_1p_1p)^2}{(pp_1p_2)}\right] + \left[(k_2^2p_2) - \frac{(k_2p_2p)^2}{(pp_1p_2)}\right] < 0. \tag{B.18}$$

Using $(pp_1p_2) < (p^2p_1)$, one has

$$(k_1^2p_1) - \frac{(k_1p_1p)^2}{(pp_1p_2)} < (k_1^2p_1) - \frac{(k_1p_1p)^2}{(p^2p_1)} \le 0, \tag{B.19}$$

where the last inequality follows from $(k_1^2p_1) - \frac{(k_1p_1p)^2}{(p^2p_1)} = (k_1)_-^2$ for $\boldsymbol{t} = \boldsymbol{p}$. The first bracket in (B.18) is thus negative. Similarly, the second bracket is negative. Thus, (B.17) holds, and the sum (1.3) is indeed absolutely convergent.

C. Details on the contact potential

C.1 Calculation of δe^Φ

There are four sources of two-instanton terms in (3.10):

- one-instanton contribution to $\mathcal{X}_\gamma$ plugged in the integral term;
- one-instanton contribution to the mirror map for u^a plugged in the central charge appearing in the same integral term;
- quadratic terms in the one-instanton contribution to the mirror map for u^a coming from first 'tree-level' term;
- two-instanton contribution to the mirror map for u^a plugged in the first term.

Collecting all these contributions together and taking the large volume limit $t^a \to \infty$, one arrives at the following result

$$\begin{aligned}
\delta e^\Phi = & -\frac{\tau_2}{16\pi^2}\sum_{\gamma\in\Gamma_+}\sigma_\gamma\bar\Omega(\gamma)\int_{\ell_\gamma}\mathrm{d}z\left[q_0 + q_ab^a + \frac{(bbp)}{2} - 2\mathrm{i}z(q_at^a + (pbt)) - \frac{3z^2(pt^2)}{2}\right]\mathcal{X}^{\mathrm{cl}}_\gamma\big(1+\mathcal{X}^{(1)}_\gamma\big) + \mathrm{c.c.}\\
& -\frac{1}{64\pi^4}\sum_{\gamma_1,\gamma_2\in\Gamma_+}\sigma_{\gamma_1}\sigma_{\gamma_2}\bar\Omega(\gamma_1)\bar\Omega(\gamma_2)(tp_1p_2)\left(\mathrm{Re}\int_{\ell_{\gamma_1}}\mathrm{d}z_1\,\mathcal{X}^{\mathrm{cl}}_{\gamma_1}\right)\left(\mathrm{Re}\int_{\ell_{\gamma_2}}\mathrm{d}z_2\,\mathcal{X}^{\mathrm{cl}}_{\gamma_2}\right).
\end{aligned} \tag{C.1}$$

To further simplify this expression, we note the following identities

$$\sum_{\gamma\in\Gamma_+}\sigma_\gamma\bar\Omega(\gamma)\int_{\ell_\gamma}\mathrm{d}z\left(\frac{1}{4\pi\tau_2}-\mathrm{i}z\left(q_a t^a+(pbt)\right)-z^2(pt^2)\right)\mathcal{X}^{\rm cl}_\gamma=0, \tag{C.2}$$

$$\sum_{\gamma_1,\gamma_2\in\Gamma_+}\sigma_{\gamma_1}\sigma_{\gamma_2}\bar\Omega(\gamma_1)\bar\Omega(\gamma_2)\int_{\ell_{\gamma_1}}\mathrm{d}z_1\int_{\ell_{\gamma_2}}\mathrm{d}z_2\,\frac{\mathrm{i}\langle\gamma_1,\gamma_2\rangle}{z_2-z_1}\left(\frac{1}{8\pi\tau_2}-\mathrm{i}z_1\left(q_{1,a}t^a+(p_1bt)\right)-z_1^2(p_1t^2)\right)\mathcal{X}^{\rm cl}_{\gamma_1}\mathcal{X}^{\rm cl}_{\gamma_2}=0.$$

The first one holds because the integrand appears to be a total derivative, while the second identity can be proven by symmetrizing in charges and integrating by parts. Then, substituting (3.23) into (C.1) and using these identities, the instanton contribution to the contact potential can be rewritten as

$$\begin{aligned}\delta e^\Phi=&-\frac{\tau_2}{16\pi^2}\sum_{\gamma\in\Gamma_+}\sigma_\gamma\bar\Omega(\gamma)\int_{\ell_\gamma}\mathrm{d}z\,\mathcal{X}^{\rm cl}_\gamma\left(\hat q_0+\frac12(\boldsymbol{q}+\boldsymbol{b})^2-\frac{\mathrm{i}z}{2}\left(q_at^a+(pbt)\right)-\frac{3}{8\pi\tau_2}\right)\\
&+\frac{\tau_2}{32\pi^3}\sum_{\gamma_1,\gamma_2\in\Gamma_+}\sigma_{\gamma_1}\sigma_{\gamma_2}\bar\Omega(\gamma_1)\bar\Omega(\gamma_2)\int_{\ell_{\gamma_1}}\mathrm{d}z_1\,\mathcal{X}^{\rm cl}_{\gamma_1}\int_{\ell_{\gamma_2}}\mathrm{d}z_2\,\mathcal{X}^{\rm cl}_{\gamma_2}\left[\frac{(tp_1p_2)}{16\pi\tau_2}\right.\\
&\left.-\left((tp_1p_2)-\frac{\mathrm{i}\langle\gamma_1,\gamma_2\rangle}{z_2-z_1}\right)\left(\hat q_{1,0}+\frac12\left(\boldsymbol{q}_1+\boldsymbol{b}\right)^2-\frac{\mathrm{i}z_1}{2}\left(q_{1,a}t^a+(p_1bt)\right)-\frac{3}{16\pi\tau_2}\right)\right]+\mathrm{c.c.}\\
&-\frac18\sum_{\boldsymbol{p}_1,\boldsymbol{p}_2}(tp_1p_2)\,\mathcal{F}^{(1)}_{\boldsymbol{p}_1}\overline{\mathcal{F}^{(1)}_{\boldsymbol{p}_2}},\end{aligned} \tag{C.3}$$

where the function $\mathcal{F}^{(1)}_{\boldsymbol{p}}$ is defined in (3.27). Using

$$\mathcal{D}_\mathfrak{h}\mathcal{X}^{\rm cl}_\gamma=-\left(\hat q_0+\frac12\left(\boldsymbol{q}+\boldsymbol{b}\right)^2-\frac{\mathrm{i}z}{2}\left(q_at^a+(pbt)\right)+\frac{\mathfrak{h}}{4\pi\tau_2}\right)\mathcal{X}^{\rm cl}_\gamma, \tag{C.4}$$

it is straightforward to check that this result is equivalent to the representation (4.5).

C.2 Calculation of the double integral

Here, we provide the details of computation of the double integral $\mathcal{Y}_{\gamma_1\gamma_2}$ defined in (4.7). We write it in the following general form

$$\mathcal{Y}_{\gamma_1\gamma_2}=\int_{\ell_{\gamma_1}}\mathrm{d}z_1\int_{\ell_{\gamma_2}}\mathrm{d}z_2\,\frac{e^{-2\pi\tau_2\left(a_1\left(z_1+\frac{\mathrm{i}b_1}{a_1}\right)^2+a_2\left(z_2+\frac{\mathrm{i}b_2}{a_2}\right)^2\right)}}{z_2-z_1}, \tag{C.5}$$

where the contours ℓ_{γ_i} in the z-plane are arcs running from -1 to 1 and passing through $-\mathrm{i}b_i/a_i$. We are interested in the limit $a_i\gg 1$, which corresponds to the large volume limit on $\mathcal{M}_H$. Then one can deform the contours into straight lines $\mathbb{R}-\mathrm{i}b_i/a_i$ since this changes the integral by an exponentially small contribution and, importantly, we do not pick up any residue while doing this. Performing the change of integration variables

$$z_1=-\frac{\mathrm{i}b_1}{a_1}+v-\frac{a_2u}{a_1+a_2},\qquad z_2=-\frac{\mathrm{i}b_2}{a_2}+v+\frac{a_1u}{a_1+a_2}, \tag{C.6}$$

one finds that the integral becomes

$$\mathcal{Y}_{\gamma_1\gamma_2}=\int_{\mathbb{R}}\mathrm{d}v\,e^{-2\pi\tau_2(a_1+a_2)v^2}\int_{\mathbb{R}}\mathrm{d}u\,\frac{e^{-2\pi\tau_2\,\frac{a_1a_2}{a_1+a_2}\,u^2}}{u+\mathrm{i}\left(\frac{b_1}{a_1}-\frac{b_2}{a_2}\right)}. \tag{C.7}$$

The first factor is Gaussian, whereas the second can be evaluated using the formula

$$\int_{\mathbb{R}} \frac{dx}{x - \mathrm{i}\alpha} e^{-\beta^2 x^2} = \mathrm{i}\pi \,\mathrm{sgn}(\mathrm{Re}(\alpha))\, e^{\alpha^2\beta^2}\, \mathrm{Erfc}(\mathrm{sgn}(\mathrm{Re}(\alpha\beta))\alpha\beta)\,. \tag{C.8}$$

For (C.7), this gives

$$\mathcal{Y}_{\gamma_1\gamma_2} = -\frac{\mathrm{i}\pi\,\mathrm{sgn}(a_2 b_1 - a_1 b_2)}{\sqrt{2\tau_2(a_1+a_2)}}\, e^{2\pi\tau_2 \frac{(a_2b_1-a_1b_2)^2}{a_1a_2(a_1+a_2)}}\, \beta_{\frac12}\left(2\tau_2\, \frac{(a_2b_1-a_1b_2)^2}{a_1a_2(a_1+a_2)}\right). \tag{C.9}$$

Substituting now $a_i = (p_i t^2)$ and $b_i = (q_{i,a} + (p_i b)_a)t^a$, and noting that $\frac{a_2b_1-a_1b_2}{\sqrt{a_1a_2(a_1+a_2)}} = \mathcal{I}_{\gamma_1\gamma_2}$, one reproduces the result (4.8).

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